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ZERO-FREE REGIONS FOR LACUNARY TYPE POLYNOMIALS

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Abstract. This paper aims to set an account of zero-free regions for lacunary type polynomials whose coefficients or their real and imaginary parts are subjected to certain restrictions. We also find bounds concerning the number of zeros in a specific annular region.

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1. Introduction

Deriving zero bounds for real and complex zeros of polynomials is a classical problem that has been proven essential in various disciplines such as engineering, mathematics, and mathematical chemistry. As indicated, there is a large body of literature dealing with the problem of providing disks in the complex plane representing so called inclusion radii (bounds) where all zeros of an univariate complex polynomial are situated. A review on the location of zeros of polynomials, where the polynomials can be factored over disks in complex plane can be found in ([13],[8],[17],[16]). In accordance with, the following first result which describes the inclusion radii where all zeros of an univariate complex polynomial are scattered is due to Cauchy [3]. All the zeros of a polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n, \quad a_n \neq 0$$

lie in the disk

$$|z| < 1 + M,$$

where $M = \max\limits_{0 \leq j \leq n-1} \left|\frac{a_j}{a_n}\right|$.
Cauchy type polynomials have been studied extensively in the past more than one-century. The research associated with this has sprawled into several directions and generates a plethora of publications for example see ([10], [12], [18], [13]). The research on mathematical objects associated with polynomials and relative position of their zeros has been active over a period; there are many research papers published in a variety of journals each year and different approaches have been taken for different purposes. The present article is concerned with zero free regions and

particularly the number of zeros of a polynomial in a given disk. The following result establishes the improvement of above Cauchy bound under the assumption that the coefficients satisfy monotonicity condition.
If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients such that

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 > 0.$$

Then $P(z)$ has all its zeros in $|z| \leq 1$. This elegant result is known as Eneström-Kakeya Theorem, (for reference see section 8.3 of [18]). In the literature, there exist various extensions and generalizations of Eneström-Kakeya Theorem ([2],[5], [6], [8], [10], [12], [13], [15], [16], [18]). Following analogous result established by Joyal et al.[10], the foremost and the most cited one after Eneström-Kakeya Theorem which acts as a generalization of it.
Let

$$a_n \geq a_{n-1} \geq a_{n-2} ... \geq a_1 \geq a_0.$$

Then the polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ of degree n has all its zeros in

$$|z| \leq \frac{1}{|a_n|}\{a_n - a_0 + |a_0|\}.$$

Several years later Aziz and Zargar [2] relaxed the hypothesis in several ways and among other things proved the following result.
Let

$$P(z) = a_0 + a_1 z + ... + a_n z^n$$

be a polynomial of degree n with real coefficients such that, for some $k \geq 1$ and for some $0 < \rho \leq 1$ we have

$$ka_n \geq a_{n-1} \geq ... \geq \rho a_0 \geq 0,$$

then $P(z)$ has all its zeros in

$$|z + k - 1| \leq k + \frac{2a_0(1-\rho)}{a_n}.$$

These results proved to be, each in its own way, enabling the growth of sophisticated techniques and critical practices are foundational in the development of the geometry of the zeros of univariate complex polynomial.

Up till now, we have precisely reviewed the regions containing all the zeros of a polynomial $P(z)$ under restricted coefficients. Since the motivation of this article is about the zero free regions and the number of zeros for special family of polynomials and in view of that it is significant to deal with some preliminary results related to zero free regions. The following result is due to Zargar [20].

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some real number $k \geq 1$

$$0 < a_n \leq a_{n-1} \leq ... \leq a_1 \leq ka_0,$$

then $P(z)$ does not vanish in the disk $|z| < \frac{1}{2k-1}$.

Generally speaking, the methods and techniques to develop the zero free and zero containing regions are different and are satisfactory for the readers. The theory on zero free regions for the univariate complex polynomials has been well established ([20], [9], [1], [4], [11]), while somewhat is known for analytic functions. This article describes zero free regions for lacunary type polynomials and this approach is new in comparison with previously published material in the study of zero free regions.

Next we move to the number of zeros of a polynomial in a given disk, the following result concerning the number of zeros of a polynomial in a closed disk can be found in Titchmarsh's classic "The Theory of Functions (see [19],page 171, 2nd edition).

Theorem 1.1. *Let $F(z)$ be analytic in $|z| \leq R$. Let $|F(z)| \leq M$ in $|z| \leq R$ and suppose $F(0) \neq 0$. Then for $0 < \delta < 1$, the number of zeros of $F(z)$ in the disk $|z| \leq R\delta$ does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|F(0)|}.$$

Regarding the number of zeros of a polynomial in $|z| \leq \frac{1}{2}$ and under the same Eneström -Kakeya type restrictions on the coefficients. Mohammad [15] used a special case of Theorem 1.1 in order to establish the following result.

Theorem 1.2. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 > 0,$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$1 + \frac{1}{\log 2} \log \frac{a_n}{a_0}.$$

This result has been refined and generalized in different ways (see [5], [7], [8], [16]). Recently Mir et al. [14] imposed certain conditions on the moduli of coefficients and among other things of the Lacunary type polynomials $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ and proved the following results.

Theorem 1.3. *Let* $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j,\ 1\le\mu<n,\ a_0\neq 0,$ *where for some* $t>0$ *and some* $\mu\le k\le n$,

$$t^{\mu}|a_{\mu}|\le ...\le t^{k-1}|a_{k-1}|\le t^k|a_k|\ge t^{k+1}a_{k+1}\ge ...\ge t^{n-1}|a_{n-1}|\ge t^n|a_n|$$

and $|arg\ a_j-\beta|\le\alpha\le\frac{\pi}{2}$ *for* $\mu\le j\le n$, *for some real* α *and* β. *Then for* $0<\delta<1$, *the number of zeros of* $P(z)$ *in* $|z|\le\delta t$ *does not exceed*

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{\mathcal{M}}{|a_0|},$$

where $\mathcal{M}=2|a_0|t+(|a_{\mu}|t^{\mu+1}+|a_n|t^{n+1})(1-\cos\alpha-\sin\alpha)+2|a_k|t^{k+1}\cos\alpha+2\sum_{j=\mu}^{n}|a_j|t^{j+1}\sin\alpha.$

Theorem 1.4. *Let* $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j,\ 1\le\mu<n,\ a_0\neq 0,$ *where Re* $a_j=\alpha_j$ *and Im* $a_j=\beta_j$ *for* $\mu\le j\le n$. *Suppose that for some* $t>0$ *and some* k *with* $\mu\le k\le n$,

$$t^{\mu}\alpha_{\mu}\le ...\le t^{k-1}\alpha_{k-1}\le t^k\alpha_k\ge t^{k+1}\alpha_{k+1}\ge ...\ge t^{n-1}\alpha_{n-1}\ge t^n\alpha_n.$$

Then for $0<\delta<1$, *the number of zeros of* $P(z)$ *in* $|z|\le\delta t$ *does not exceed*

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{\mathcal{M}}{|a_0|},$$

where $\mathcal{M}=2(|\alpha_0|+|\beta_0|)t+(|\alpha_{\mu}|-\alpha_{\mu})t^{\mu+1}+2\alpha_kt^{k+1}+(|\alpha_n|-\alpha_n)t^{n+1}+2\sum_{j=\mu}^{n}|\beta_j|t^{j+1}.$

Theorem 1.5. *Let* $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j,\ 1\le\mu<n,\ a_0\neq 0,$ *where Re* $a_j=\alpha_j$ *and Im* $a_j=\beta_j$ *for* $\mu\le j\le n$. *Suppose that for some* $t>0$ *,for some* k *with* $\mu\le k\le n$, *we have*

$$t^{\mu}\alpha_{\mu}\le ...\le t^{k-1}\alpha_{k-1}\le t^k\alpha_k\ge t^{k+1}\alpha_{k+1}\ge ...\ge t^{n-1}\alpha_{n-1}\ge t^n\alpha_n,$$

and for some $\mu\le l\le n$, *we have*

$$t^{\mu}\beta_{\mu}\le ...\le t^{l-1}\alpha_{l-1}\le t^l\beta_l\ge t^{l+1}\beta_{l+1}\ge ...\ge t^{n-1}\beta_{n-1}\ge t^n\beta_n.$$

Then for $0<\delta<1$, *the number of zeros of* $P(z)$ *in* $|z|\le\delta t$ *does not exceed*

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{\mathcal{M}}{|a_0|},$$

where

$$\mathcal{M}=2(|\alpha_0|+|\beta_0|)t+(|\alpha_{\mu}|-\alpha_{\mu}+|\beta_{\mu}|-\beta_{\mu})t^{\mu+1}+$$
$$+2(\alpha_kt^{k+1}+\beta_lt^{l+1})+(|\alpha_n|-\alpha_n+|\beta_{\mu}|-\beta_{\mu})t^{n+1}.$$

2. Main results

The purpose of this paper is to obtain zero free regions for the lacunary type polynomials whose coefficients satisfy certain monotonicity conditions. We shall also establish the annular region so that number of zeros of $P(z)$ in this region does not exceed any given real number. Also the parameters can be adapted appropriately to the intensity required. In fact we prove the following results.

Theorem 2.1. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu < n,\ a_0 \neq 0,$ *where for some* $t > 0$ *and some* $\mu \le k \le n,$

$$t^{\mu}|a_{\mu}| \le ... \le t^{k-1}|a_{k-1}| \le t^k|a_k| \ge t^{k+1}|a_{k+1}| \ge ... \ge t^{n-1}|a_{n-1}| \ge t^n|a_n|$$

and $|arg\ a_j - \beta| \le \alpha \le \frac{\pi}{2}$ *for* $\mu \le j \le n$*, for some real* α *and* β*. Then no zero of* $P(z)$ *lies in*

$$|z| < \frac{t^2|a_0|}{|a_0|t + (|a_{\mu}|t^{\mu+1} + |a_n|t^{n+1})(1 - \sin\alpha - \cos\alpha) + 2|a_k|t^{k+1}\cos\alpha + 2\sum_{j=\mu}^{n}|a_j|t^{j+1}\sin\alpha}.$$

Theorem 2.1 in conjunction with Theorem 1.3, immediately leads to the following result.

Corollary 2.1. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu < n,\ a_0 \neq 0,$ *where for some* $t > 0$ *and some* $\mu \le k \le n,$

$$t^{\mu}|a_{\mu}| \le ... \le t^{k-1}|a_{k-1}| \le t^k|a_k| \ge t^{k+1}a_{k+1} \ge ... \ge t^{n-1}|a_{n-1}| \ge t^n|a_n|$$

and $|arg\ a_j - \beta| \le \alpha \le \frac{\pi}{2}$ *for* $\mu \le j \le n$*, for some real* α *and* β*. Then for* $0 < \delta < 1$*, the number of zeros of* $P(z)$ *in* $\frac{t^2|a_0|}{\mathcal{M}_1} \le |z| \le \delta t$ *does not exceed*

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{\mathcal{M}}{|a_0|},$$

where

$$\mathcal{M} = 2|a_0|t + (|a_{\mu}|t^{\mu+1} + |a_n|t^{n+1})(1 - \cos\alpha - \sin\alpha) + 2|a_k|t^{k+1}\cos\alpha + 2\sum_{j=\mu}^{n}|a_j|t^{j+1}\sin\alpha$$

$$\mathcal{M}_1 = |a_0|t + (|a_{\mu}|t^{\mu+1} + |a_n|t^{n+1})(1 - \sin\alpha - \cos\alpha) + 2|a_k|t^{k+1}\cos\alpha + 2\sum_{j=\mu}^{n}|a_j|t^{j+1}\sin\alpha.$$

Notice that when $t = 1$ in Theorem 2.1, it produces the following result.

Corollary 2.2. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu < n,\ a_0 \neq 0,$ *where for some* $\mu \le k \le n.$

Theorem 2.2. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu < n,\ a_0 \ne 0,$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \le j \le n$. *Suppose that for some* $t > 0$ *and some* k *with* $\mu \le k \le n$,

$$t^{\mu}\alpha_{\mu} \le ... \le t^{k-1}\alpha_{k-1} \le t^k\alpha_k \ge t^{k+1}\alpha_{k+1} \ge ... \ge t^{n-1}\alpha_{n-1} \ge t^n\alpha_n.$$

Then no zero of $P(z)$ *lies in*

$$|z| < \frac{t^2(|\alpha_0| + |\beta_0|)}{(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu)t^{\mu+1} + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + 2\sum_{j=\mu}^{n}|\beta_j| t^{j+1}}.$$

On combining Theorem 2.2 and Theorem 1.4, we get the following result.

Corollary 2.3. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu < n,\ a_0 \ne 0,$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \le j \le n$. *Suppose that for some* $t > 0$ *and some* k *with* $\mu \le k \le n$, *we have*

$$t^{\mu}\alpha_{\mu} \le ... \le t^{k-1}\alpha_{k-1} \le t^k\alpha_k \ge t^{k+1}\alpha_{k+1} \ge ... \ge t^{n-1}\alpha_{n-1} \ge t^n\alpha_n.$$

Then for $0 < \delta < 1$, *the number of zeros of* $P(z)$ *in* $\frac{t^2(|\alpha_0|+|\beta_0|)}{\mathcal{M}_2} \le |z| \le \delta t$ *does not exceed*

$$\frac{1}{\log\frac{1}{\delta}} \log \frac{\mathcal{M}}{|a_0|},$$

where

$$\mathcal{M} = 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu)t^{\mu+1} + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + 2\sum_{j=\mu}^{n}|\beta_j| t^{j+1}$$

and

$$\mathcal{M}_2 = (|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu)t^{\mu+1} + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + 2\sum_{j=\mu}^{n}|\beta_j| t^{j+1}.$$

Taking $t = 1$ in Theorem 2.2, we get the following result.

Corollary 2.4. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu < n,\ a_0 \ne 0,$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \le j \le n$. *Suppose that for some* $\mu \le k \le n$, *we have*

$$\alpha_{\mu} \le ... \le \alpha_{k-1} \le \alpha_k \ge \alpha_{k+1} \ge ... \ge \alpha_{n-1} \ge \alpha_n.$$

Then $P(z)$ *does not vanish in*

$$|z| < \frac{(|\alpha_0| + |\beta_0|)}{(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu) + 2\alpha_k + (|\alpha_n| - \alpha_n) + 2\sum_{j=\mu}^{n}|\beta_j|}.$$

Finally, we put the monotonicity conditions on the real and imaginary parts of the coefficients of $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ in order to obtain zero free region and an annular region onwards. More precisely, we prove the following results.

Theorem 2.3. *Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu < n,\ a_0 \neq 0,$ where Re $a_j = \alpha_j$ and Im $a_j = \beta_j$ for $\mu \leq j \leq n$. Suppose that for some $t > 0$,for some k with $\mu \leq k \leq n$,*

$$t^{\mu}\alpha_{\mu} \leq ... \leq t^{k-1}\alpha_{k-1} \leq t^{k}\alpha_{k} \geq t^{k+1}\alpha_{k+1} \geq ... \geq t^{n-1}\alpha_{n-1} \geq t^{n}\alpha_{n}$$

and for some $\mu \leq l \leq n$,

$$t^{\mu}\beta_{\mu} \leq ... \leq t^{l-1}\alpha_{l-1} \leq t^{l}\beta_{l} \geq t^{l+1}\beta_{l+1} \geq ... \geq t^{n-1}\beta_{n-1} \geq t^{n}\beta_{n}.$$

Then $P(z)$ does not vanish in

$$|z| < \frac{t^2(|\alpha_0| + |\beta_0|)}{(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu)t^{\mu+1} + 2(\alpha_k t^{k+1} + \beta_l t^{l+1}) + k\, t^{n+1}}.$$

where $k = |\alpha_n| - \alpha_n + |\beta_n| - \beta_n$.

Theorem 2.3 in conjunction with Theorem 1.5 yields the following result.

Corollary 2.5. *Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu < n,\ a_0 \neq 0,$ where Re $a_j = \alpha_j$ and Im $a_j = \beta_j$ for $\mu \leq j \leq n$. Suppose that for some $t > 0$,for some k with $\mu \leq k \leq n$,*

$$t^{\mu}\alpha_{\mu} \leq ... \leq t^{k-1}\alpha_{k-1} \leq t^{k}\alpha_{k} \geq t^{k+1}\alpha_{k+1} \geq ... \geq t^{n-1}\alpha_{n-1} \geq t^{n}\alpha_{n}$$

and for some $\mu \leq l \leq n$,

$$t^{\mu}\beta_{\mu} \leq ... \leq t^{l-1}\alpha_{l-1} \leq t^{l}\beta_{l} \geq t^{l+1}\beta_{l+1} \geq ... \geq t^{n-1}\beta_{n-1} \geq t^{n}\beta_{n}.$$

Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in $\frac{t^2(|\alpha_0|+|\beta_0|)}{\mathcal{M}_3} \leq |z| \leq \delta t$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}} \log \frac{\mathcal{M}}{|a_0|},$$

where

$$\begin{aligned}\mathcal{M} = 2(|\alpha_0| + |\beta_0|)t &+ (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu)t^{\mu+1} \\ &+ 2(\alpha_k t^{k+1} + \beta_l t^{l+1}) + (|\alpha_n| - \alpha_n + |\beta_\mu| - \beta_\mu)t^{n+1},\end{aligned}$$

$$\begin{aligned}\mathcal{M}_3 = (|\alpha_0| + |\beta_0|)t &+ (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu)t^{\mu+1} \\ &+ 2(\alpha_k t^{k+1} + \beta_l t^{l+1}) + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n)t^{n+1}.\end{aligned}$$

Here it is interesting to note that Theorem 2.3 gives us several corollaries under the monotonicity conditions on real and imaginary parts. Taking $t = 1$ in Theorem 2.3, we get the following result.

Corollary 2.6. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu < n,\ a \ne 0,$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \le j \le n$. *Suppose that for some* k *with* $\mu \le k \le n$, *we have*

$$\alpha_\mu \le ... \le \alpha_{k-1} \le \alpha_k \ge \alpha_{k+1} \ge ... \ge \alpha_{n-1} \ge \alpha_n$$

and for some $\mu \le l \le n$,

$$\beta_\mu \le ... \le \alpha_{l-1} \le \beta_l \ge \beta_{l+1} \ge ... \ge \beta_{n-1} \ge \beta_n.$$

Then no zero of $P(z)$ *lies in*

$$|z| < \frac{(|\alpha_0| + |\beta_0|)}{(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu) + 2(\alpha_k + \beta_l) + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n)}.$$

Fix $t = 1$ and $k = l = n$ in Theorem 2.3, we immediately obtain the following result.

Corollary 2.7. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu < n,\ a_0 \ne 0,$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \le j \le n$ *such that*

$$\alpha_\mu \le ... \le \alpha_{n-1} \le \alpha_n$$

and

$$\beta_\mu \le ... \le \beta_{n-1} \le \beta_n.$$

Then no zero of $P(z)$ *lies in*

$$|z| < \frac{(|\alpha_0| + |\beta_0|)}{(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu) + (|\alpha_n| + \alpha_n + |\beta_\mu| + \beta_\mu)}.$$

Set $t = 1$ and $k = l = \mu$ in Theorem 2.3, we get the following result.

Corollary 2.8. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu < n,\ a_0 \ne 0,$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \le j \le n$ *such that*

$$\alpha_\mu \ge ... \ge \alpha_{n-1} \ge \alpha_n$$

and

$$\beta_\mu \ge ... \ge \beta_{n-1} \ge \beta_n.$$

Then no zero of $P(z)$ *lies in*

$$|z| < \frac{(|\alpha_0| + |\beta_0|)}{(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| + \alpha_\mu + |\beta_\mu| + \beta_\mu) + (|\alpha_n| - \alpha_n + |\beta_\mu| - \beta_\mu)}.$$

3. Proofs of theorems

For the proofs of our main results, we need the following auxiliary result.

Lemma 3.1. *Let $P(z)$ be a polynomial of degree n. If for some real α and β, $|arg\ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, $0 \leq j \leq n$ and for any $t > 0$ such that, $|a_j| \geq |a_{j-1}|, 0 \leq j \leq n$, then $\big|ta_j - a_{j-1}\big| \leq \big(t|a_j| - |a_{j-1}|\big)cos\alpha + \big(t|a_j| + |a_{j-1}|\big)sin\alpha$.*

The above lemma is due to Govil and Rahman [8].

Proof of Theorem 2.1 Consider the polynomial

$$F(z) = (t-z)P(z) = (t-z)\bigg(a_0 + \sum_{j=\mu}^{n} a_j z^j\bigg).$$

This implies,

$$F(z) = a_0 t + \sum_{j=\mu}^{n} ta_j z^j - a_0 z - \sum_{j=\mu}^{n} a_j z^{j+1} = a_0(t-z) + \sum_{j=\mu}^{n} ta_j z^j - \sum_{j=\mu+1}^{n+1} a_{j-1} z^j$$

i.e., $F(z) = a_0(t-z) + ta_\mu z^\mu + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j - a_n z^{n+1} = a_0 t + R(z)$, where $R(z) = -a_0 z + ta_\mu z^\mu + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j - a_n z^{n+1}$. On $|z| = t$, we have

$$\begin{aligned}|R(z)| =& \bigg| -a_0 z + ta_\mu z^\mu + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j - a_n z^{n+1}\bigg| \\ \leq& |a_0|t + |a_\mu| t^{\mu+1} + \sum_{j=\mu+1}^{n} |ta_j - a_{j-1}| t^j + |a_n| t^{n+1}.\end{aligned}$$

Equivalently,

$$|R(z)| \leq t|a_0| + |a_\mu| t^{\mu+1} + \sum_{j=\mu+1}^{k} |ta_j - a_{j-1}| t^j + \sum_{j=k+1}^{n} |ta_j - a_{j-1}| t^j + |a_n| t^{n+1}.$$

Using lemma 3.1, we get

$$\begin{aligned}|R(z)| \leq t|a_0| + |a_\mu| t^{\mu+1} + &\sum_{j=\mu+1}^{k} \{(|a_j|t - |a_{j-1}|)\cos\alpha + (|a_j|t + |a_{j-1}|)\sin\alpha\} t^j \\ &+ \sum_{j=k+1}^{n} \{(|a_{j-1}| - |a_j|t)\cos\alpha + (|a_j|t + |a_{j-1}|)\sin\alpha\} t^j + |a_n| t^{n+1} \\ &= t|a_0| + |a_\mu| t^{\mu+1} + \sum_{j=\mu+1}^{k} |a_j| t^{j+1}\cos\alpha - \sum_{j=\mu+1}^{k} |a_{j-1}| t^j \cos\alpha \\ &+ \sum_{j=\mu+1}^{k} |a_j| t^{j+1}\sin\alpha + \sum_{j=\mu+1}^{k} |a_{j-1}| t^j \sin\alpha + \sum_{j=k+1}^{n} |a_{j-1}| t^j \cos\alpha\end{aligned}$$

$$- \sum_{j=k+1}^{n} |a_j| t^{j+1} \cos\alpha + \sum_{j=k+1}^{n} |a_{j-1}| t^{j} \sin\alpha + \sum_{j=k+1}^{n} |a_j| t^{j+1} \sin\alpha + |a_n| t^{n+1}.$$

This gives,

$$|R(z)| \le |a_0|t + |a_\mu|t^{\mu+1} - |a_\mu|t^{\mu+1}\cos\alpha + |a_k|t^{k+1}\cos\alpha + |a_\mu|t^{\mu+1}\sin\alpha$$

$$+ |a_k|t^{k+1}\sin\alpha + 2\sum_{j=\mu+1}^{k-1} |a_j|t^{j+1} + |a_k|t^{k+1}\cos\alpha - |a_n|t^{n+1}\cos\alpha + |a_k|t^{k+1}\sin\alpha$$

$$+ |a_n|t^{n+1}\sin\alpha + 2\sum_{j=k+1}^{n-1} |a_j|t^{j+1}\sin\alpha + |a_n|t^{n+1} = |a_0|t + (|a_\mu|t^{\mu+1} + |a_n|t^{n+1})$$

$$\times (1 - \sin\alpha - \cos\alpha) + 2|a_k|t^{k+1}\cos\alpha + 2\sum_{j=\mu}^{n} |a_j|t^{j+1}\sin\alpha = \mathcal{M}_1.$$

Applying Schwarz lemma to $R(z)$, we get $|R(z)| \le \dfrac{\mathcal{M}_1|z|}{t}, \quad |z| \le t$. Hence

$$|F(z)| = |a_0 t + R(z)| \ge |a_0|t - |R(z)| \ge |a_0|t - \frac{\mathcal{M}_1|z|}{t} > 0 \qquad for\ |z| \le t,$$

if $|a_0|t - \dfrac{\mathcal{M}_1|z|}{t} > 0$. That is, if $|z| < \dfrac{t^2|a_0|}{\mathcal{M}_1}$. This shows that $F(z)$ and hence $P(z)$ has no zero in $|z| < \frac{t^2|a_0|}{\mathcal{M}_1}$. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2 We consider

$$F(z) = (t-z)P(z) = a_0(t-z) + ta_\mu z^\mu + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j - a_n z^{n+1}.$$

Equivalently,

$$F(z) = (\alpha_0 + i\beta_0)(t-z) + (\alpha_\mu + i\beta_\mu)tz^\mu + \sum_{j=\mu+1}^{n} (\alpha_j t - \alpha_{j-1})z^j$$

$$+ i\sum_{j=\mu+1}^{n} (\beta_j t - \beta_{j-1})z^j - (\alpha_n + i\beta_n)z^{n+1} = (\alpha_0 + i\beta_0)t + R(z).$$

For $|z| = t$, we have

$$|R(z)| \le (|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^{n} |\alpha_j t - \alpha_{j-1}|t^j$$

$$+ \sum_{j=\mu+1}^{n} |\beta_j t - \beta_{j-1}|t^j + (|\alpha_n| + |\beta_n|)t^{n+1}$$

$$= (|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^{k} (\alpha_j t - \alpha_{j-1})t^j$$

$$+ \sum_{j=k+1}^{n} (\alpha_{j-1} - \alpha_j t)t^j + \sum_{j=\mu+1}^{n} \big(|\beta_j|t + |\beta_{j-1}|\big)t^j + (|\alpha_n| + |\beta_n|)t^{n+1} =$$

$$= (|\alpha_0|+|\beta_0|)t+(|\alpha_\mu|-\alpha_\mu)t^{\mu+1}+2\alpha_k t^{k+1}+(|\alpha_n|-\alpha_n)t^{n+1}+2\sum_{j=\mu}^{n}|\beta_j|t^{j+1} = \mathcal{M}_2.$$

Applying Schwarz lemma to the polynomial $R(z)$, we get

$$|R(z)| \le \frac{\mathcal{M}_2|z|}{t}, \qquad for\ |z| \le t.$$

Hence $|F(z)| = |a_0 t + R(z)| \ge |a_0|t - |R(z)| \ge |a_0|t - \dfrac{\mathcal{M}_2|z|}{t} > 0$, $|z| \le t$, if $|a_0|t - \dfrac{\mathcal{M}_2|z|}{t} > 0$, that is, if $|z| < \dfrac{t^2|a_0|}{\mathcal{M}_2}$. This shows that $F(z)$ and hence $P(z)$ has no zero in $|z| < \frac{t^2|a_0|}{\mathcal{M}_2}$ and the proof of Theorem 2.2 is complete. □

Proof of Theorem 2.3 As in the proof of Theorem 2.2,

$$\begin{aligned} F(z) =&(\alpha_0+i\beta_0)(t-z)+(\alpha_\mu+i\beta_\mu)tz^\mu+\sum_{j=\mu+1}^{n}(\alpha_j t-\alpha_{j-1})z^j \\ &+i\sum_{j=\mu+1}^{n}(\beta_j t-\beta_{j-1})z^j-(\alpha_n+i\beta_n)z^{n+1} = (\alpha_0+i\beta_0)t+R(z). \end{aligned}$$

where

$$\begin{aligned} R(z) =&-(\alpha_0+i\beta_0)z+(\alpha_\mu+i\beta_\mu)tz^\mu+\sum_{j=\mu+1}^{n}(\alpha_j t-\alpha_{j-1})z^j \\ &+i\sum_{j=\mu+1}^{n}(\beta_j t-\beta_{j-1})z^j-(\alpha_n+i\beta_n)z^{n+1}. \end{aligned}$$

For $|z| = t$, we have

$$\begin{aligned} |R(z)| \le& (|\alpha_0|+|\beta_0|)t+(|\alpha_\mu|+|\beta_\mu|)t^{\mu+1}+\sum_{j=\mu+1}^{n}|\alpha_j t-\alpha_{j-1}|t^j \\ &+\sum_{j=\mu+1}^{n}|\beta_j t-\beta_{j-1}|t^j+(|\alpha_n|+|\beta_n|)t^{n+1} = (|\alpha_0|+|\beta_0|)t+(|\alpha_\mu|+|\beta_\mu|)t^{\mu+1} \\ &+\sum_{j=\mu+1}^{k}(\alpha_j t-\alpha_{j-1})t^j+\sum_{j=k+1}^{n}(\alpha_{j-1}-\alpha_j t)t^j+\sum_{j=\mu+1}^{k}(\beta_j t-\beta_{j-1})t^j \\ &+\sum_{j=k+1}^{n}(\beta_{j-1}-\beta_j t)t^j+(|\alpha_n|+|\beta_n|)t^{n+1} = (|\alpha_0|+|\beta_0|)t \\ &+(|\alpha_\mu|-\alpha_\mu+|\beta_\mu|-\beta_\mu)t^{\mu+1}+2(\alpha_k t^{k+1}+\beta_l t^{l+1}) \\ &+(|\alpha_n|-\alpha_n+|\beta_n|-\beta_n)t^{n+1} = \mathcal{M}_3. \end{aligned}$$

Applying Schwarz lemma to the polynomial $R(z)$, we get

$$|R(z)| \le \frac{\mathcal{M}_2|z|}{t}, \qquad for\ |z| \le t.$$

Hence

$$
\begin{aligned}
|F(z)| =& |(\alpha_0 + i\beta_0)t + R(z)| \geq (|\alpha_0| + |\beta_0|)t - |R(z)| \\
&\geq (|\alpha_0| + |\beta_0|)t - \frac{\mathcal{M}_3|z|}{t} > 0, \qquad |z| \leq t,
\end{aligned}
$$

if

$$
(|\alpha_0| + |\beta_0|)t - \frac{\mathcal{M}_3|z|}{t} > 0.
$$

That is, if

$$
|z| < \frac{(|\alpha_0| + |\beta_0|)t^2}{\mathcal{M}_3}.
$$

This shows that $F(z)$ and hence $P(z)$ has no zero in $|z| < \frac{(|\alpha_0|+|\beta_0|)t^2}{\mathcal{M}_3}$. This completes the proof of Theorem 2.3. □

4. Examples

Since the present article is concerned with newly developed approach to obtain the zero free regions and the number of zeros for the lacunary type polynomials in a given disk. From this point of view, the comparison of the bounds obtained with the previous bounds appropriately have no scope within this type of study. Instead of comparing the bounds, we point out few examples which may be helpful to be examined.

Example 4.1. Let $P(z) = 2z^5 + 2.5z^4 + 4z^3 + 3z^2 + 2z + 1$. Clearly, here $\mu = 1$ and $n = 5$. We take $k = 3$, $\alpha = \pi/2$ and $t = 1$. In view of Theorem 2.1 and due to this type of intensity of parameters the radius of given disk comes out to be 0.0357. Since the appropriate zeros of $P(z)$ are $-0.358+0.9154i$, $-0.358-0.9154i$, $0.0756+0.8657i$, $0.0756-0.8657i$, -0.6853. Then one can see that $P(z)$ does not vanish in $|z| < 0.0357$.

Since corollary 2.1.1 is the union of Theorem 2.1 and Theorem 1.3. Under the same example it is clear that all the zeros of $P(z) = 2z^5 + 2.5z^4 + 4z^3 + 3z^2 + 2z + 1$ lie in $|z| \geq 0.0357$. If we set $\delta = 0.7$, the upper bound of the annular region in corollary 2.1.1 comes out to be 0.7 as $t = 1$. In this case, we found that the number of zeros of underlying polynomial $P(z)$ in $0.0357 \leq |z| \leq 0.7$ does not exceed $\frac{1}{\log \frac{1}{0.7}} \log(29) \approx 9.4524$. Hence we conclude that $P(z)$ has at most one zero in $0.0357 \leq |z| \leq 0.7$ and of course, $P(z)$ has exactly one zero in $0.0357 \leq |z| \leq 0.7$. All above discussion demonstrates one thing, which is beauty to say, that the bound in Theorem 2.1 becomes the lower bound of the annular region in corollary 2.1.1.

Example 4.2. Let $P(z) = 2z^5 + 3z^4 + 4z^3 + 2z^2 + 1.5z + 1$. Clearly, here $\mu = 1$ and $n = 5$. Setting $k = 3$ and $t = 1$. In view of Theorem 2.2 the radius comes out to be

0.1111. Numerically the appropriate zeros of $P(z)$ are $-0.6193+1.0343i,\ -0.6193-1.0343i,\ 0.2089+0.6804i,\ 0.2089-0.6804i,\ 0.6792$. It is clear from these zeros that $P(z)$ does not vanish in $|z|<0.1111$.

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