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COVARIOGRAM AND ORIENTATION-DEPENDENT CHORD LENGHT DISTRIBUTION FUNCTION FOR OBLIQUE PRISM

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Abstract. In the present paper we obtain explicit form of covariogram and oriented-dependent chord length distribution function for oblique prism when we know the covariogram of base. Additional we calculate the covariogram and oriented-dependent chord length distribution function in the case if the base is any trapezoid.

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1. Introduction

Blaschke formulated the question whether the chord length distribution function characterizes a set [15]. The answer to this question is negative. Mallows and Clark presented non-congruent convex polygons with the same chord length distribution function[11]. There are many articles ([6],[7],[16]) where for solving this problem it is considered that a subclass of the class of convex bodies for which the chord length distribution function is not equal for non-congruent members.

A convex body in R^n is a compact convex set K with non empty interior. Denote by L_n n-dimensional Lebesgue measure on R^n. If $x \in R^n$, D+x denote the translate of D by x, i.e.,

$$D + x = \{y + x, y \in D\}$$

If $D \subset R^n$ is a convex body, then its covariogram $C_D(x)$ is the function defined for $x \in R^n$ by

$$C_D(x) = L_n(D \cap (D + x)).$$

G. Matheron posed in [12] the following question.

Covariogram Problem. Does the covariogram determine a convex body D in R^n, among all convex bodies, up to translation and reflection?

Reflection in this paper always means reflection at a point. Matheron problem is true if n=2 and it is false for $n \geq 4$, but for $n = 3$ it is still open. In [12] Matheron showed that for every $t > 0$ and $\phi \in S^{n-1}$ (S^{n-1} is (n-1)-dimensional unit sphere centered at the origin)

$$\frac{\partial C_D(t\phi)}{\partial t} = -L_{n-1}(\{y \in \phi^\perp : L_1(D \cap (l_\phi + y)) \geq t\}) \tag{1.1}$$

where $l_\phi + y$ denotes the line parallel to ϕ through the point y, while $\phi^\perp$ denotes the hyperplane in R^n with normal direction $\phi \in S^{n-1}$.

Let G be the space of all lines in the Euclidean plane R^2, $g \in G$ and (p, ϕ) is the polar coordinates of the foot of the perpendicular to g from the origin, $p \geq 0$, $\phi \in S^1$. For a closed bounded convex domain $D \subset R^2$ we denote by $S_D(\phi)$ the support function in direction $\phi \in S^1$ defined by

$$S_D(\phi) = max\{p \geq 0 : g(p, \phi) \cap D \neq \emptyset\}$$

For a bounded convex domain $D \subset R^2$ we denote by $b_D(\phi)$ the breadth function in direction $\phi \in S^1$, that is, the distance between two support lines to the boundary of D that are perpendicular to ϕ. We have

$$b_D(\phi) = S_D(\phi) + S_D(\phi + \pi)$$

Note that $b_D(\phi)$ is a periodic function with period π [15].

For a bounded convex domain D the chord length distribution function in direction ϕ, denoted by $F_D(x, \phi)$, is defined to be the probability of having chord $\chi(g) = g \cap D$ with length at most x in the bundle of lines parallel to ϕ. A random line which is parallel to ϕ and intersects D has an intersection point (denoted by y) with the line $\phi^\perp$. The intersection point y is uniformly distributed on the segment $[0, b_D(\phi)]$. Thus, we have

$$F_D(x, \phi) = \frac{L_1(y \in \Pi_D(\phi) : \chi(l_\phi + y) \leq x)}{b_D(\phi)} \tag{1.2}$$

It is not difficult to verify that for n = 2 formula (1.1) is equivalent to

$$\frac{\partial C_D(t\phi)}{\partial t} = -b_D(\phi)(1 - F_D(t, \phi)) \tag{1.3}$$

Denote by Γ the space of lines γ in R^3. Let $\Pi_D(\omega)$ denote the projection of a bounded convex body $D \subset R^3$ in direction $\omega \in S^2$ and let $s_D(\omega)$ be its area. Every line which is parallel to ω and intersects D has an intersection with $\Pi_D(\omega)$. Denote that point by y and that line by $l_\omega + y$. The intersection point y is uniformly distributed on $\Pi_D(\omega)$. The chord length distribution function of D in direction $\omega \in S^2$ is defined by

$$F_D(x, \omega) = \frac{L_2\{y : \chi(l_\omega + y) \leq x\}}{s_D(\omega)} \tag{1.4}$$

It is easy to verify that for n = 3 formula (1.1) is equivalent to

$$\frac{\partial C_D(t\phi)}{\partial t} = -s_D(\phi)(1 - F_D(t,\phi)) \tag{1.5}$$

This article aims to calculate covariogram and orientation-dependent chord length distribution functions (see [1],[2],[5],[8],[9],[13],[14]).

In this paper, we obtain the following results

1)The calculation of the covariogram and Orientation-dependent chord length distribution function for any trapezoid. This is a generalization of the result of [14].

2)Relationships between the covariogram and the orientation-dependent chord length distribution function of an oblique prism and those of its base.

3) Explicit forms of the covariogram and the orientation-dependent chord length distribution function of an oblique prism with cyclic, elliptical, trapezoid and triangular bases. The second and third results are a generalization of [9].

2. Computation of chord length distribution function of an oblique prism

Consider the oblique prism U with base B (not necessarily convex), the length of prism generator is equal to d and angle between prism generator and base is equal to β. It is obvious that the domain $U \cap (U + x)$ is also a prism. If we denote by t the length of x and by $\omega = (\phi, \theta)$, (ϕ, θ) is the cylindrical parametrization of ω; $\phi \in S_1, \theta \in [-\pi/2, \pi/2]$ the direction of x, then the base of the prism $U \cap (U + x)$ will be the domain $B \cap (B + y)$, where y is a planar vector of length $\frac{t\sin(\beta-\theta)}{\sin\beta}$ and direction ϕ, and the height of the new prism will be $d\sin\beta - t\sin\theta$(due to the symmetry we consider only the case $\theta \in [0, \pi/2]$). We can say that

$$C_U(x) = C_U(x\omega) = L_3(U \cap (U+t\omega)) = L_2(B \cap (B+\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right)\phi)(d\sin\beta - t\sin\theta)$$

Implying that

$$C_U(t\omega) = C_B(\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right)\phi)(d\sin\beta - t\sin\theta) \tag{2.1}$$

Differentiating both sides of equation (2.1) with respect to t, we get

(2.2)

$$\frac{\partial C_U(t\omega)}{\partial t} = -\sin\theta C_B(\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right)\phi) + (d\sin\beta - t\sin\theta)\frac{\partial C_B(\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right)\phi)}{\partial t}$$

Using equation (1.3)

$$-\frac{\partial C_B(\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right)\phi)}{\partial t} = b_B(\phi)\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right)(1 - F_B(\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right), \phi)) \tag{2.3}$$

If we integrate both parts of equation (1.3) from 0 to $\frac{t\sin(\beta-\theta)}{\sin\beta}$, we get

(2.4)

$$C_B(\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right)\phi) = ||B|| - b_B(\phi)\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right)\int_0^t (1-F_B(\left(\frac{u\sin(\beta-\theta)}{\sin\beta}\right),\phi))du$$

where $||B||$ is the area of B. Using equations (1.5),(2.3),(2.4) we can transform equation (2.2) the following way

$$s_U(\omega)(1-F_U(t,\omega)) = \sin\theta(||B|| - b_B(\phi)\left(\frac{\sin(\beta-\theta)}{\sin\beta}\right)\int_0^t (1-F_B(\left(\frac{u\sin(\beta-\theta)}{\sin\beta}\right),\phi))du) +$$

$$+(d\sin\beta - t\sin\theta)b_B(\phi)\left(\frac{\sin(\beta-\theta)}{\sin\beta}\right)(1-F_B(\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right),\phi))$$

We can say that

$$s_U(\omega) = ||B||\sin\theta + b_B(\phi)d\sin\beta\left(\frac{\sin(\beta-\theta)}{\sin\beta}\right)$$

Using above mentioned we can formulate the following theorem

Theorem 2.1. *For oblique prism U with base B (not necessarily convex), with prism generator d and angle between prism generator and base β the orineted-dependent chord length distribution given by the following formula*

(2.5)

$$F_U(t,\omega) = \begin{cases} 0, & \text{if } t \le 0 \\ \dfrac{b_B(\phi)\left(\frac{\sin(\beta-\theta)}{\sin\beta}\right)}{||B||\sin\theta + b_B(\phi)d\sin\beta\left(\frac{\sin(\beta-\theta)}{\sin\beta}\right)} \times & \\ \times\Big(t\sin\theta + \sin\theta\int_0^t (1-F_B(\left(\frac{u\sin(\beta-\theta)}{\sin\beta}\right),\phi))du + & \\ +(d\sin\beta - t\sin\theta)F_B(\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right),\phi)\Big), & \text{if } 0 \le t \le t_{max}(\omega) \\ 1, & \text{if } t \ge t_{max}(\omega) \end{cases}$$

Where $t_{max}(\omega)$ is

(2.6)

$$t_{max}(\omega) = \begin{cases} \frac{\sin\beta x_{max}(\phi)}{|\sin(\beta-\theta)|}, & \text{if } \theta \in [-\arctan\frac{d\sin\beta}{x_{max}(\phi)-d\sin\beta}, \arccos\frac{d\sin\beta}{d\sin\beta + x_{max}(\phi)}] \\ \frac{d\sin\beta}{|\sin\theta|}, & \text{otherwise} \end{cases}$$

when $d\cos\beta < x_{max}(\phi)$ and

(2.7)

$$t_{max}(\omega) = \begin{cases} \frac{\sin\beta x_{max}(\phi)}{|\sin(\beta-\theta)|}, & \text{if } \theta \in [0, \arctan\frac{d\sin\beta}{x_{max}(\phi)+d\sin\beta}] \\ \frac{d\sin\beta}{|\sin\theta|}, & \text{if } \theta \in [\arctan\frac{d\sin\beta}{x_{max}(\phi)+d\sin\beta}, \arctan\frac{d\sin\beta}{d\sin\beta - x_{max}(\phi)}] \\ \frac{\sin\beta x_{max}(\phi)}{|\sin(\theta-\beta)|}, & \text{if } \theta \in [\arctan\frac{d\sin\beta}{d\sin\beta - x_{max}(\phi)}], \pi/2] \cup [-\pi/2, 0] \end{cases}$$

when $d\cos\beta > x_{max}(\phi)$.

3. Chord length distribution in a trapezoid

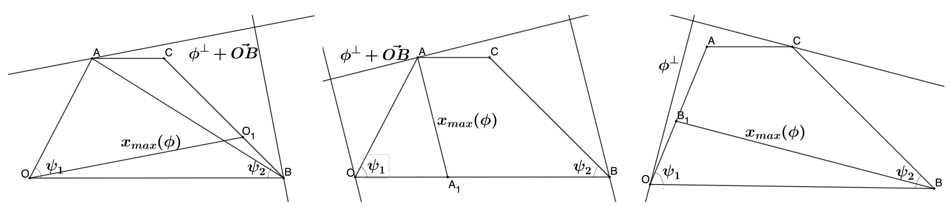
Let $T \subset R^2$ be a trapezoid with bases a and b and the angle between longer base and legs are ψ_1, ψ_2. Without loss of generality we can assume that $0 < \psi_1 \leq \pi/2$, $\psi_1 \leq \psi_2 < \pi$ and $b \leq a$. We can translate and rotate trapezoid so that the longer base be on X-axis.

It is obvious that, the height of trapezoid is equal to $h = (a-b)\frac{sin\psi_1 sin\psi_2}{sin(\psi_1+\psi_2)}$, the side OA is equal to $l_{OA} = (a-b)\frac{sin\psi_2}{sin(\psi_1+\psi_2)}$ and the side CB is equal to $l_{CB} = (a-b)\frac{sin\psi_1}{sin(\psi_1+\psi_2)}$. From here we can say that the vertices of trapezoid are O(0,0), A($(a-b)\frac{cos\psi_1 sin\psi_2}{sin(\psi_1+\psi_2)}$,$(a-b)\frac{sin\psi_1 sin\psi_2}{sin(\psi_1+\psi_2)}$),B(a,0),C(b+$(a-b)\frac{cos\psi_1 sin\psi_2}{sin(\psi_1+\psi_2)}$,$(a-b)\frac{sin\psi_1 sin\psi_2}{sin(\psi_1+\psi_2)}$). If we take the square or rectangle we should know height and side instead of the above mentioned quantities.

For calculating the orientation-dependent chord length distribution function of a trapezoid, we firstly need explicit form of breadth function of the trapezoid.

Lemma 3.1. *Let $T \subset R^2$ be trapezoid with bases a and b and the angle between longer base and legs are ψ_1 ψ_2. We can assume that the longer leg is equal to a and $\psi_1 \leq \psi_2$. Then the breadth function has the following form*

$$b_T(\phi) = \begin{cases} l_{CB}sin(\phi+\psi_2) + bsin\phi, & \text{if } 0 \leq \phi \leq \psi_1 \\ asin\phi, & \text{if } \psi_1 \leq \phi \leq \pi-\psi_2 \\ bsin(\phi) + l_{OA}sin(\phi-\psi_1), & \text{if } \pi-\psi_2 \leq \phi \leq \pi \end{cases} \tag{3.1}$$



(a) $0 \leq \phi \leq \psi_1$ (b) $\psi_1 \leq \phi \leq \pi-\psi_2$ (c) $\pi-\psi_2 \leq \phi \leq \pi$

Рис. 1

Proof. To prove this lemma firstly we should understand which two vertices have the last intersection with lines in direction ϕ. This means that we should find the $l_\phi + y$ for every Vertex and take the two vertices for which y has the minimum and the maximum value.

(Case i) for $0 \leq \phi < \psi_1$ two vertices are A and B. That means the $b_T(\phi)$ is equal to the projection of AB diagonal onto $\phi^\perp$.

$$b_T(\phi) = L_1(\Pi_{AB}(\phi)) = l_{CB}sin(\phi+\psi_2) + bsin\phi$$

(Case ii) for $\psi_1 \leq \phi < \pi - \psi_2$ two vertices are O and B. That means the $b_T(\phi)$ is equal to the projection of OB base onto $\phi^{\perp}$.

$$b_T(\phi) = L_1(\Pi_{OB}(\phi)) = asin\phi$$

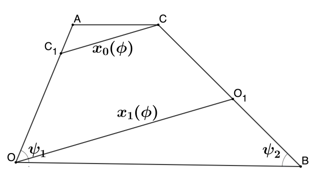
(Case iii) for $\pi - \psi_2 \leq \phi < \pi$ two vertices are C and O. That means the $b_T(\phi)$ is equal to the projection of OC diagonal onto $\phi^{\perp}$.

$$b_T(\phi) = L_1(\Pi_{CO}(\phi)) = bsin(\phi) + l_{OA}sin(\phi - \psi_1). \qquad \square$$

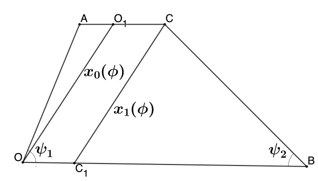
We denote the lines $x_0(\phi)$ and $x_1(\phi)$ which has ϕ angle with X-axis, pass through a vertex of trapezoid and make a chord of positive Lebesgue measure,

$$x_0(\phi) = min\chi(l_\phi + y) \text{ and } x_1(\phi) = max\chi(l_\phi + y)$$

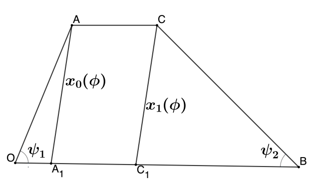
Figure 2 shows all cases of above mentioned quantities.



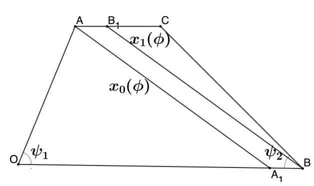
(a) $\pi k \leq \phi < \pi k + \arcsin \frac{h}{l_{OC}}$



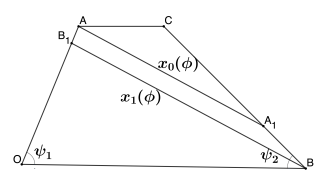
(b) $\pi k + \arcsin \frac{h}{l_{OC}} \leq \phi <$ $< \pi k + \psi_1$



(c) $\psi_1 + \pi k \leq \phi < \pi(k+1) -$ $-\psi_2$



(d) $\pi(k+1) - \psi_2 \leq \phi < \pi(k+1) -$ $- \arcsin \frac{h}{l_{AB}}$



(e) $\pi(k+1) - \arcsin \frac{h}{l_{AB}} \leq \phi <$ $< \pi(k+1)$

Рис. 2

Lemma 3.2. *$x_1(\phi) = x_{max}(\phi)$ for any angle ϕ. If we choose some $k \in Z$ we should have the following cases for $x_0(\phi)$ and $x_1(\phi)$*

(i) If $\pi k \leq \phi < \psi_1 + \pi k$

$$x_0(\phi) = \begin{cases} \frac{b \sin \psi_1}{|\sin(\psi_1 - \phi)|}, & \text{if } \pi k \leq \phi < \pi k + \arcsin \frac{h}{l_{OC}} \\ \frac{h}{|\sin \phi|}, & if \pi k + \arcsin \frac{h}{l_{OC}} \leq \phi < \pi k + \psi_1 \end{cases}$$

$$x_{max}(\phi) = \begin{cases} \frac{a \sin \psi_2}{|\sin(\psi_2 + \phi)|}, & \text{if } \pi k \leq \phi < \pi k + \arcsin \frac{h}{l_{OC}} \\ \frac{h}{|\sin \phi|}, & if \pi k + \arcsin \frac{h}{l_{OC}} \leq \phi < \pi k + \psi_1 \end{cases}$$

(ii) If $\psi_1 + \pi k \leq \phi < \pi(k+1) - \psi_2$

$$x_0(\phi) = x_1(\phi) = \frac{h}{|\sin\phi|}$$

(iii) If $\pi(k+1) - \psi_2 \leq \phi < \pi(k+1)$ *, and* $l_{OA}\sin\psi_1 < a$

$$x_0(\phi) = \begin{cases} \frac{h}{|\sin\phi|}, & if\ \pi(k+1) - \psi_2 \leq \phi < \pi(k+1) - \arcsin\frac{h}{l_{AB}} \\ \frac{b\sin\psi_2}{|\sin(\phi+\psi_2)|}, & if \pi(k+1) - \arcsin\frac{h}{l_{AB}} \leq \phi < \pi(k+1) \end{cases}$$

$$x_{max}(\phi) = \begin{cases} \frac{h}{|\sin(\phi)|}, & if\ \pi(k+1) - \psi_2 \leq \phi < \pi(k+1) - \arcsin\frac{h}{l_{AB}} \\ \frac{a\sin\psi_1}{|\sin(\phi-\psi_1)|}, & if \pi(k+1) - \arcsin\frac{h}{l_{AB}} \leq \phi < \pi(k+1) \end{cases}$$

(iv) If $\pi(k+1) - \psi_2 \leq \phi < \pi(k+1)$ *, and* $l_{OA}\cos\psi_1 > a$

$$x_0(\phi) = \begin{cases} \frac{h}{|\sin\phi|}, & if\ \pi(k+1) - \psi_2 \leq \phi < \arcsin\frac{h}{l_{AB}} \\ \frac{b\sin\psi_2}{|\sin(\phi+\psi_2)|}, & if\ \arcsin\frac{h}{l_{AB}} \leq \phi < \pi(k+1) \end{cases}$$

$$x_{max}(\phi) = \begin{cases} \frac{h}{|\sin(\phi)|}, & if\ \pi(k+1) - \psi_2 \leq \phi < \arcsin\frac{h}{l_{AB}} \\ \frac{a\sin\psi_1}{|\sin(\phi-\psi_1)|}, & if\ \arcsin\frac{h}{l_{AB}} \leq \phi < \pi(k+1) \end{cases}$$

Proof. A chord of maximal length in a convex polygon with direction ϕ, also known as ϕ-diameter of the polygon, is not necessarily unique but for any given ϕ exists a ϕ-diameter such that at least one endpoint of the chord coincides with a vertex of the given polygon.

Case (i) sub-case 1 ($\pi k \leq \phi < \pi k + \arcsin\frac{h}{l_{OC}}$) From Figure 2a it can be seen that $x_0(\phi) = CC_1$ and $x_1(\phi) = x_{max}(\phi) = OO_1$. By Sine Rule

$$x_0(\phi) = \frac{b\sin(180 - \psi_1)}{\sin(\psi_1 - \phi + \pi k)} = \frac{b\sin\psi_1}{|\sin(\psi_1 - \phi)|}$$

$$x_1(\phi) = x_{max}(\phi) = \frac{a\sin\psi_2}{\sin(180 - \psi_2 - \phi + \pi k)} = \frac{a\sin\psi_2}{|\sin(\psi_2 + \phi)|}$$

Case (i) sub-case 2 ($\pi k + \arcsin\frac{h}{l_{OC}} \leq \phi < \pi k + \psi_1$) From Figure 2b it shows that $x_0(\phi) = x_1(\phi) = x_{max}(\phi) = CC_1$. By Sine Rule

$$x_0(\phi) = x_1(\phi) = x_{max}(\phi) = \frac{h}{|\sin\phi|}$$

Case (ii) ($\psi_1 + \pi k \leq \phi < \pi(k+1) - \psi_2$) From Figure 2c it can be seen that $x_0(\phi) = x_1(\phi) = x_{max}(\phi) = CC_1$. By Sine Rule

$$x_0(\phi) = x_1(\phi) = x_{max}(\phi) = \frac{h}{|\sin\phi|}$$

Case (iii) sub-case 1 ($\pi(k+1) - \psi_2 \leq \phi < \pi(k+1) - \arcsin\frac{h}{l_{AB}}$) From Figure 2d it shows that $x_0(\phi) = BB_1$ and $x_1(\phi) = x_{max}(\phi) = AA_1$. By Sine Rule

$$x_0(\phi) = \frac{l_{CB}\sin(180 - \psi_2)}{\sin(180 - \phi + \pi k)} = \frac{h}{|\sin\phi|}$$

$$x_1(\phi) = x_{max}(\phi) = \frac{h}{\sin(180 - \phi + \pi k)} = \frac{h}{|\sin\phi|}$$

Case (iii) sub-case 2 ($\pi(k+1) - \arcsin\frac{h}{l_{AB}} \le \phi < \pi(k+1)$) From Figure 2e it can be seen that $x_0(\phi) = AA_1$ and $x_1(\phi) = x_{max}(\phi) = BB_1$. By Sine Rule

$$x_0(\phi) = \frac{b\sin(180-\psi_2)}{\sin(\phi+\psi_2-180+\pi k)} = \frac{b\sin\psi_2}{|\sin(\phi+\psi_2)|}$$

$$x_1(\phi) = x_{max}(\phi) = \frac{a\sin\psi_1}{|\sin(\phi-\psi_1)|}$$

The proof of case (iv) has the same steps as case(iii). □

Theorem 3.1. *$F_T(x,\phi) = 0$ if $x < 0$ and $F_T(x,\phi) = 1$ if $x > x_{max}(\phi)$.Now we discuss the non-trivial cases when $0 < x < x_{max}$. Because this is π periodic function we can assume that k is equal to 0.*

(i) For $0 \le \phi < \psi_1$

$$F_T(x,\phi) = \begin{cases} \frac{x\sin\phi(\sin(\psi_1-\phi)\sin\psi_2+\sin\psi_1\sin(\phi+\psi_2))}{b_T(\phi)\sin\psi_1\sin\psi_2}, & if\ 0 \le x < x_0(\phi) \\ \frac{1}{b_T(\phi)}(b\sin\phi + \frac{(x-x_0(\phi))\sin(\psi_1-\phi)\sin(\psi_2+\phi)}{\sin(\psi_1+\psi_2)} + \\ +\frac{x\sin(\phi+\psi_2)\sin\phi}{\sin\psi_2}), & if x_0(\phi) \le x < x_{max}(\phi) \end{cases}$$

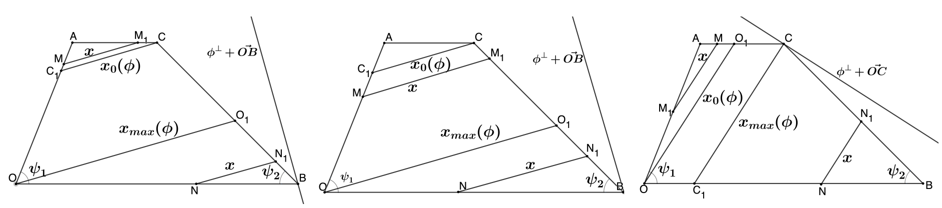
(ii) For $\psi_1 \le \phi < \pi - \psi_2$

$$F_T(x,\phi) = \frac{x\sin\phi}{b_T(\phi)}\left(\frac{\sin(\psi_2+\phi)\sin\psi_1 + \sin(\phi-\psi_1)\sin\psi_2}{\sin\psi_1\sin\psi_2}\right)$$

(iii) For $\pi - \psi_2 \le \phi < \pi$

$$F_T(x,\phi) = \begin{cases} \frac{-x\sin\phi(\sin\psi_1\sin(\phi+\psi_2)-\sin(\phi-\psi_1)\sin\psi_2)}{b_T(\phi)\sin\psi_1\sin\psi_2}, & if\ 0 \le x \le x_0(\phi) \\ \frac{1}{b_T(\phi)}(b\sin\phi - \frac{(x-x_0(\phi))sin(\psi_2+\phi)\sin(\phi-\psi_1)}{\sin(\psi_1+\psi_2)} + \\ +\frac{x\sin\phi\sin(\phi-\psi_1)}{\sin\psi_1}), & if x_0(\phi) \le x < x_{max}(\phi) \end{cases}$$

Proof.



(a) $0 \le \phi < \arcsin\frac{h}{l_{OC}}$, $0 \le x < x_0(\phi)$

(b) $0 \le \phi < \arcsin\frac{h}{l_{OC}}$, $x_0(\phi) \le x < x_{max}(\phi)$

(c) $\arcsin\frac{h}{l_{OC}} \le \phi < \psi_1$, $0 \le x < x_{max}(\phi)$

Рис. 3

Case (i) sub-case 1 let $0 \le \phi < \arcsin\frac{h}{l_{OC}}$ and $0 \le x < x_0(\phi)$. In Figure 3a $|MM_1| = |NN_1| = x < x_0(\phi) = |CC_1| < |OO_1| = x_{max}(\phi)$. For this we can say that $F_T(x,\phi) = \frac{1}{b_T(\phi)}(b_{\Delta AMM_1}(\phi) + b_{\Delta BNN_1}(\phi))$. Here $b_{\Delta AMM_1}(\phi)$ and $b_{\Delta BNN_1}(\phi)$

are equal to the height of triangle AMM_1 (with base MM_1) and BNN_1 (with base NN_1)

$$b_{\Delta AMM_1}(\phi) = \frac{x\sin(\psi_1-\phi)\sin\phi}{sin\psi_1}$$

$$b_{\Delta BNN_1}(\phi) = \frac{x\sin(\psi_2+\phi)\sin\phi}{sin\psi_2}$$

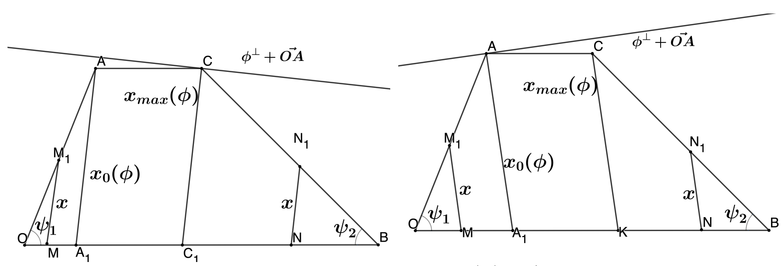
Case (i) sub-case 2 let $0 \le \phi < \arcsin\frac{h}{l_{OC}}$ and $x_0(\phi) \le x < x_{max}(\phi)$. In Figure 3b $x_0(\phi) = CC_1 < x = MM_1 = NN_1 < x_{max}(\phi)$. In this case we have $F(x,\phi) = \frac{1}{b_T(\phi)}(b_{ACMM_1}(\phi) + b_{\Delta BNN_1}(\phi)) = \frac{1}{b_T(\phi)}(b_{ACC_1}(\phi) + b_{\Delta BNN_1}(\phi)) + b_{MCC_1M_1}) = \frac{1}{b_T(\phi)}(b\sin\phi + \frac{x\sin(\psi_2+\phi)\sin\phi}{sin\psi_2} + b_{MCC_1M_1})$. We should calculate the height of trapezoid MCC_1M_1

$$b_{MCC_1M_1} = \frac{\sin(\psi_1-\phi)\sin(\psi_2+\phi)(x-x_0(\phi))}{\sin(\psi_1+\psi_2)}$$

Case (i) sub-case 3 let $\arcsin\frac{h}{l_{OC}} \le \phi < \psi_1$ and $0 \le x < x_{max}(\phi)$. In Figure 3c $x = |NN_1| = |MM_1| < |CC_1| = |OO_1| = x_0(\phi) = x_{max}(\phi)$. Computations of this case are identical as in the previous case (1) sub-case 1. Completing the above we can say that for any $\phi \in [0,\psi_1]$ it brings to

$$F_T(x,\phi) = \begin{cases} \frac{1}{b_T(\phi)}(\frac{x\sin(\psi_1-\phi)\sin\phi}{sin\psi_1} + \frac{x\sin(\psi_2+\phi)\sin\phi}{sin\psi_2}), & \text{if } 0 \le x < x_0(\phi) \\ \frac{1}{b_T(\phi)}(b\sin\phi + \frac{\sin(\psi_1-\phi)\sin(\psi_2+\phi)(x-x_0(\phi)}{\sin(\psi_1+\psi_2)} + \\ + \frac{x\sin(\phi+\psi_2)\sin\phi}{\sin\psi_2}), & if x_0(\phi) \le x < x_{max}(\phi) \end{cases}$$

Case (ii) sub-case 1 $\psi_1 \le \phi < \pi/2$ and $0 \le x < x_{max}(\phi)$. Here $F_t(x,\psi) =$



(a) $\psi_1 \le \phi < \pi/2$, $0 \le x < x_{max}(\phi)$

(b) $\pi/2 \le \phi < \pi - \psi_2$, $0 \le x < x_{max}(\phi)$

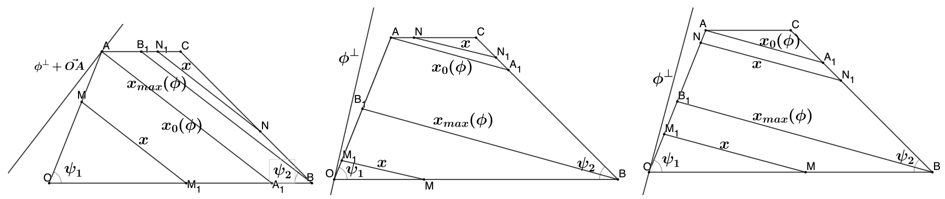
Рис. 4

$\frac{1}{b_T(\phi)}(\frac{x\sin(\phi-\psi_1)\sin\phi}{sin\psi_1} + \frac{x\sin(\psi_2+\phi)\sin\phi}{sin\psi_2})$.

Case (ii) Sub-case 2 $\pi/2 \le \phi < \pi - \psi_2$ and $0 \le x < x_{max}(\phi)$. In Figure 4b $x = |NN_1| = |MM_1| < |CC_1| = |AA_1| = x_0(\phi) = x_{max}(\phi)$ we have

$$F_T(x,\phi) = \frac{1}{b_T(\phi)}(b_{\Delta OMM_1} + B_{\Delta BNN_1}) =$$

$$= \frac{1}{b_T(\phi)}(\frac{x\sin(\phi-\psi_1)\sin\phi}{sin\psi_1} + \frac{x\sin(\psi_2+\phi)\sin\phi}{sin\psi_2})$$



(a) $\pi-\psi_2 \le \phi < \pi-\psi_2 - \arcsin\frac{h}{l_{AB}}$, $0 \le x \le x_{max}(\phi)$

(b) $\pi-\psi_2-\arcsin\frac{h}{l_{AB}} \le \phi < \pi$, $0 \le x < x_0(\phi)$

(c) $\pi-\psi_2-\arcsin\frac{h}{l_{AB}} \le \phi < \pi$, $x_0(\phi) \le x < x_{max}(\phi)$

Рис. 5

Case (iii) sub-case 1 $\pi-\psi_2 \le \phi < \pi-\psi_2-\arcsin\frac{h}{l_{AB}}$ In Figure 5a $x=|NN_1|=|MM_1|<|AA_1|=|BB_1|=x_0(\phi)=x_{max}(\phi)$

$$F_t(x,\phi)=\frac{1}{b_T(\phi)}(b_{\Delta OMM_1}+b_{\Delta CNN_1})=$$

$$=\frac{1}{b_T(\phi)}(\frac{-x\sin(\psi_2+\phi)\sin\phi}{\sin\psi_2}+\frac{x\sin(\phi-\psi_1)\sin\phi}{\sin\psi_1})$$

Case (iii) sub-case 2 $\pi-\psi_2-\arcsin\frac{h}{l_{AB}} \le \phi < \pi$ and $0 \le x < x_0(\phi)$. In Figure 5b $x=|MM_1|=|NN_1|=x<|AA_1|=x_0(\phi)<|BB_1|=x_{max}(\phi)$

$$F_T(x,\phi)=\frac{1}{b_T(\phi)}(b_{\Delta CNN_1}(\phi)+b_{\Delta OMM_1}(\phi))=$$

$$=\frac{1}{b_T(\phi)}(\frac{-x\sin(\psi_2+\phi)\sin\phi}{\sin\psi_2}+\frac{x\sin(\phi-\psi_1)\sin\phi}{\sin\psi_1})$$

Case (iii) sub-case 3 $\pi-\psi_2-\arcsin\frac{h}{l_{AB}} \le \phi < \pi$ and $x_0(\phi) \le x < x_{max}(\phi)$

$$F_T(x,\phi)=\frac{1}{b_T(\phi)}(b_{ACNN_1}+b_{\Delta OMM_1})=\frac{1}{b_T(\phi)}(b_{\Delta CAA_1}+b_{AN_1A_1N}+b_{\Delta OMM_1})$$

$$=\frac{1}{b_T(\phi)}(b\sin\phi-\frac{(x-x_0(\phi))sin(\psi_2+\phi)\sin(\phi-\psi_1)}{\sin(\psi_1+\psi_2)}+\frac{x\sin\phi\sin(\phi-\psi_1)}{\sin\psi_1}). \qquad \square$$

Object	The angles ψ_1,ψ_2	The basis a, b and height h	Article
Square	$\psi_1=\psi_2=\pi/2$	a=b=h	[13]
Rectangle	$\psi_1=\psi_2=\pi/2$	a=b $\neq h$	[14]
Parallelogram	$\psi_1=\pi-\psi_2$	a=b	[4]
Right trapezoid	$\psi_2=\pi/2$	a>b	[14]

We can use theorem 3.1 and obtain the known results of orientation-dependent chord length distribution function (for square and rectangle instead of two bases we should know height and one base). In the table above we show how to do that.

4. Computation of covariogram and chord length distribution function of oblique prism

4.1. **The case of a cyclic oblique prism.** Let L_r be an oblique prism with radius (of the base) r, side d and sides lean over at the base is β. The covariogram of a disc with radius r is

$$C_r(t,\phi)=\begin{cases}2r^2\arccos\frac{t}{2r}-\frac{t}{2}\sqrt{4r^2-t^2}, & \text{if } 0\le t\le 2r\\ 0, & \text{otherwise}\end{cases}$$

Using equation (2.1) for the covariogram L_r we obtain

$$C_{L_r}(t,\omega)=\begin{cases}(d\sin\beta-t\sin\theta)2r^2\arccos\frac{t\sin(\theta-\beta)}{2r\sin\beta}-\\ -\frac{t\sin(\theta-\beta)}{2\sin\beta}\sqrt{4r^2-\frac{t\sin(\theta-\beta)}{\sin\beta}}, & \text{if } 0\le t\le\chi_{max}(\omega)\\ 0, & \text{otherwise}\end{cases}$$

where $\chi_{max}(\omega)$ we calculate using (2.6) or (2.7)

For the orientation-dependent chord length distribution function we have

$$F_r(t,\phi)=\begin{cases}0, & \text{if } t<0\\ 1-\sqrt{1-\frac{t^2}{4r^2}}, & \text{if } 0\le t<2r\\ 1, & \text{if } t\ge 2r\end{cases}$$

Using equation (2.5) and knowing that $\chi_{max}(\phi)=2r$, we obtain

$$F_{L_r}(t,\phi)=\begin{cases}0, & \text{if } t<0\\ \frac{2\left(\frac{\sin(\beta-\theta)}{\sin\beta}\right)}{\pi r\sin\theta+2d\sin(\beta-\theta)}\Bigg(d\sin\beta-(d\sin\beta-\\ -\frac{3t\sin\theta)}{2})\left(\sqrt{1-\left(\frac{t\sin(\beta-\theta)}{2r\sin\beta}\right)^2}\right)+\\ +\frac{r\sin\theta\sin\beta}{\sin(\theta-\beta)}(\arcsin(\frac{t\sin(\beta-\theta)}{2r\sin\beta})) & \text{if } 0\le t<\chi_{max}(\omega)\\ 1, & \text{if } t\ge\chi_{max}(\omega)\end{cases}$$

4.2. **The case of an elliptic oblique prism.** Consider a prism L_e with prism generator d, the angle with prism generator and base is β and base as an ellipse with semi-major axes a and b. The covariogram of an ellipse with semi-major axes a and b has the form [10]:

$$C_r(t,\phi)=\begin{cases}2ab\left(\frac{\pi}{2}-\frac{t}{\chi_{max}(\phi)}\sqrt{1-\frac{t^2}{\chi_{max}(\phi)}}-\arcsin\frac{t}{\chi_{max}(\phi)}\right), & \text{if } 0\le t<\chi_0(\phi)\\ 0, & \text{otherwise}\end{cases}$$

where

$$\chi_{max}(\phi)=\frac{2ab}{\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}}$$

is the maximum chord in direction ϕ.

From (2.1) we get

$$C_{L_r}(t\omega) = 2ab\Bigg(\frac{\pi}{2} - \frac{t\sin(\beta-\theta)}{\chi_{max}(\phi)\sin\beta}\sqrt{1-\frac{t^2\sin^2(\beta-\theta)}{\chi_{max}(\phi)}\sin^2\beta} -$$

$$\text{-arcsin}\, \frac{t\sin(\beta-\theta)}{\chi_{max}(\phi)\sin\beta}\Bigg)\Bigg(d\sin\beta - t\sin\theta\Bigg)$$

where $\chi_{max}(\omega)$ we can calculate using equation (2.6) or (2.7)

For the orientation-dependent chord length distribution function we have [10].

$$F_e(t,\phi) = \begin{cases} 0, & \text{if } t < 0 \\ 1 - \sqrt{1 - \frac{t^2}{\chi_{max}(\phi)}}, & \text{if } 0 \le t < \chi_{max}(\phi) \\ 1, & \text{if } t \ge \chi_{max}(\phi) \end{cases}$$

Using equation (2.5) we get

$$F_{L_e}(t,\phi) = \begin{cases} 0, & \text{if } t < 0 \\ \frac{b_e(\phi)\left(\frac{\sin(\beta-\theta)}{\sin\beta}\right)}{\pi ab\sin\theta + b_e(\phi) d\sin(\beta-\theta)}\Bigg(d\sin\beta - (d\sin\beta - & \\ -\frac{3t\sin\theta)}{2})\Bigg(\sqrt{1-\left(\frac{t\sin(\beta-\theta)}{\chi_{max}(\phi)\sin\beta}\right)^2}\Bigg) + & \\ +\frac{\chi_{max}(\phi)\sin\theta\sin\beta}{2\sin(\theta-\beta)}(\arcsin(\frac{t\sin(\beta-\theta)}{\chi_{max}(\phi)\sin\beta})) & \text{if } 0 \le t < \chi_{max}(\omega) \\ 1, & \text{if } t \ge \chi_{max}(\omega) \end{cases}$$

and $b_e(\phi)$ is equal to

$$b_e(\phi) = \sqrt{a^2\sin^2\phi + b^2\cos^2\phi}$$

4.3. **The case of a triangle oblique prism.** Let L_Δ denote an oblique prism with triangular base Δ. We consider the side of Δ that lies on the X axes. Let a be the length of that side, and ψ_1 and ψ_2 be the corresponding adjacent angles. In [3] it is shown that the covariogram of Δ is given by

$$C_\Delta(t,\phi) = \begin{cases} S_\Delta\left(1 - \frac{t}{\chi_{max}(\phi)}\right)^2, & \text{if } 0 \le t < \chi_{max}(\phi) \\ 0, & \text{otherwise} \end{cases}$$

where S_Δ is the area of the triangle Δ, while $\chi_{max}(\phi)$ is defined by the following formula

$$\chi_{max}(\phi) = \begin{cases} a\sin\psi_2, & \text{if } 0 \le \phi < \psi_1 \\ a\sin\psi_1\sin\psi_2, & \text{if } \psi_1 \le \phi < \pi - \psi_2 \\ a\sin\psi_1, & \text{if } \pi - \psi_2 \le \phi < \pi \end{cases}$$

Taking into account (2.1), we obtain

$$C_{L_\Delta}(t,\phi) = \begin{cases} S_\Delta\left(1 - \frac{t\sin(\beta-\theta)}{\sin\beta\chi_{max}(\phi)}\right)^2\Bigg(s\sin\beta - t\sin\theta\Bigg), & \text{if } 0 \le t < \chi_{max}(\omega) \\ 0, & \text{otherwise} \end{cases}$$

where $\chi_{max}(\phi)$ is defined by (2.6) or (2.7). Again from [3] we have

$$F_\Delta(t,\phi) = \begin{cases} 0, & \text{if } t < 0 \\ \frac{t}{\chi_{max}(\phi)}, & \text{if } 0 \le t < \chi_{max}(\phi) \\ 1, & \text{if } t \ge \chi_{max}(\phi) \end{cases}$$

Using equation (2.5) we get

$$(4.1)\quad F_U(t,\omega) = \begin{cases} 0, & \text{if } t \le 0 \\ \frac{b_B(\phi)t\left(\frac{\sin(\beta-\theta)}{\sin\beta}\right)}{S_\Delta \sin\theta + b_B(\phi) d \sin\beta\left(\frac{2\sin(\beta-\theta)}{\sin\beta}\right)} \times & \\ \times\left(2\sin\theta - \frac{3tsin\theta\sin(\beta-\theta)}{2\sin\beta\chi_{max}(\phi)} + \frac{d\sin(\beta-\theta)}{\chi_{max}(\phi)}\right), & \text{if } 0 \le t \le t_{max}(\omega) \\ 1, & \text{if } t \ge t_{max}(\omega) \end{cases}$$

If for the three sub-sections above we take $\beta = \pi/2$ then we have same results as in [9].

4.4. **The case of a trapezoidal oblique prism.** Denote by D_T the oblique prism with tapezoidal base Using Matheron's formula we can say that

$$\frac{\partial C_T(t,\phi)}{\partial t} = -b_T(\phi)(1 - F_T(t,\phi))$$

If we integrate both parts the last equations yields

$$(4.2)\qquad C_T(t,\phi) = C_T(0,\phi) - b_T(\phi)\int_0^t (1 - F_T(u,\phi))du$$

Using equation (2.1) and Theorem 3.1 we come to explicit formula for $C_T(\phi)$. It is enough to compute for $\phi \in [0,\pi]$ because $C(\cdot,\phi)$ is π-periodic function.

$$C_T(t,\phi) = \frac{h(a+b)}{2} - tb_T(\phi) + b_T(\phi)\int_0^t F_T(u,\phi)du = \frac{h(a+b)}{2} - tb_T(\phi) +$$

$$\begin{cases} \frac{t^2\sin\phi(\sin(\psi_1-\phi)\sin\psi_2 + \sin\psi_1\sin(\phi+\psi_2))}{2\sin\psi_1\sin\psi_2}, & \text{if } 0 \le \phi \le \psi_1,\ 0 \le t < x_0(\phi) \\ tb\sin\phi + \frac{t^2\sin(\psi_1-\phi)\sin(\psi_2+\phi)}{2\sin(\psi_1+\psi_2)} - & \\ -\frac{tx_0(\phi)\sin(\psi_1-\phi)\sin(\psi_2+\phi)}{\sin(\psi_1+\psi_2)} + \frac{t^2\sin(\phi+\psi_2)\sin\phi}{2\sin\psi_2}, & \text{if } 0 \le \phi \le \psi_1,\ x_0(\phi) \le t < x_{max}(\phi) \\ t^2\sin\phi\left(\frac{\sin(\psi_2+\phi)\sin\psi_1 + \sin(\phi-\psi_1)\sin\psi_2}{2\sin\psi_1\sin\psi_2}\right), & \text{if } \psi_1 \le \phi \le \pi-\psi_2,\ 0 \le t \le t_{max}(\phi) \\ \frac{-t^2\sin\phi(\sin\psi_1\sin(\phi+\psi_2) - \sin(\phi-\psi_1)\sin\psi_2)}{2\sin\psi_1\sin\psi_2}, & \text{if } \pi-\psi_2 \le \phi \le \pi,\ 0 \le t < x_0(\phi) \\ tb\sin\phi - \frac{t^2 sin(\psi_2+\phi)\sin(\phi-\psi_1)}{2\sin(\psi_1+\psi_2)} + & \\ +\frac{tx_0(\phi)sin(\psi_2+\phi)\sin(\phi-\psi_1)}{\sin(\psi_1+\psi_2)} + \frac{t^2\sin\phi\sin(\phi-\psi_1)}{2\sin\psi_1}, & \text{if } \pi-\psi_2 \le \phi \le \pi,\ x_0(\phi) \le x < x_{max}(\phi) \end{cases}$$

Using equation (2.5) we can find explicit form of orientation-dependent chord length distribution function of oblique prism with trapezoid base.

Denote by

$$m_1(\phi) = \frac{\sin\phi(\sin(\psi_1-\phi)\sin\psi_2 + \sin\psi_1\sin(\phi+\psi_2))}{b_T(\phi)\sin\psi_1\sin\psi_2},$$

$$c_1(\phi) = \frac{1}{b_T(\phi)}(b\sin\phi - \frac{x_0(\phi)(\sin(\psi_1-\phi)\sin(\psi_2+\phi)}{\sin(\psi_1+\psi_2)})$$

$$m_2(\phi) = \frac{1}{b_T(\phi)}(\frac{(\sin(\psi_1-\phi)\sin(\psi_2+\phi)}{\sin(\psi_1+\psi_2)} + \frac{\sin(\phi+\psi_2)\sin\phi}{\sin\psi_2})$$

$$m_3(\phi) = \frac{\sin\phi}{b_T(\phi)}\left(\frac{\sin(\psi_2+\phi)\sin\psi_1 + \sin(\phi-\psi_1)\sin\psi_2}{\sin\psi_1\sin\psi_2}\right)$$

$$m_4(\phi) = \frac{-\sin\phi(\sin\psi_1\sin(\phi+\psi_2) - \sin(\phi-\psi_1)\sin\psi_2)}{b_T(\phi)\sin\psi_1\sin\psi_2}$$

$$c_2(\phi) = \frac{1}{b_T(\phi)}(b\sin\phi + \frac{x_0(\phi)sin(\psi_2+\phi)\sin(\phi-\psi_1)}{\sin(\psi_1+\psi_2)})$$

$$m_5(\phi) = \frac{1}{b_T(\phi)}(-\frac{sin(\psi_2+\phi)\sin(\phi-\psi_1)}{\sin(\psi_1+\psi_2)} + \frac{\sin\phi\sin(\phi-\psi_1)}{\sin\psi_1})$$

Using the notations above we can rewrite Theorem 3.1

Theorem 3.1(rewrite) $F_T(x,\phi) = 0$ if $x < 0$ and $F_T(x,\phi) = 1$ if $x > x_{max}(\phi)$. Now we discuss the non-trivial cases when $0 < x < x_{max}(\phi)$. Because this is π periodic function we can assume that k is equal to 0.

(i) For $0 \le \phi < \psi_1$

$$F_T(x,\phi) = \begin{cases} xm_1(\phi), & \text{if } 0 \le x < x_0(\phi) \\ xm_2(\phi) + c_1(\phi), & \text{if } x_0(\phi) \le x < x_{max}(\phi) \end{cases}$$

(ii) For $\psi_1 \le \phi < \pi - \psi_2$

$$F_T(x,\phi) = xm_3(\phi)$$

(iii)For $\pi - \psi_2 \le \phi < \pi$

$$F_T(x,\phi) = \begin{cases} xm_4(\phi), & \text{if } 0 \le x \le x_0(\phi) \\ xm_5(\phi) + c_2(\phi), & \text{if } x_0(\phi) \le x < x_{max}(\phi) \end{cases}$$

Lemma 4.1. *For oblique prism with trapezoid base we have chord length distribution function as (for shortness denote by $c = \frac{\sin(\beta-\theta)}{\sin\beta}$)*

(i) If $\pi k \le \phi \le \psi_1 + \pi k$ and $x_0(\phi) \ge \frac{x_{max}(\omega)|\sin(\beta-\theta)|}{\sin\beta}$

$$F_{D_t}(t,\omega) = \frac{b_B(\phi)c}{||B||\sin\theta + b_B(\phi)d\sin\beta c}$$

$$\left(2t\sin\theta + (d\sin\beta - t\sin\theta)tcm_1(\phi) - \frac{t^2c\sin\theta m_1(\phi)}{2}\right)$$

(ii) If $\pi(k+1) - \psi_2 \le \phi \le \pi(k+1)$ and $x_0(\phi) \le \frac{x_{max}(\omega)|\sin(\beta-\theta)|}{\sin\beta}$

For this case we have 2 sub-cases for calculating $F_T(uc,\phi)$

$$F_T(uc,\phi) = \begin{cases} um_1(\phi)c, & \text{if } u < \frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|} \\ um_2(\phi)c + c_1(\phi), & \text{if } \frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|} \le u \le \chi_{max}(\omega) \end{cases}$$

Therefore we get

$$F_{D_t}(t,\omega) = \frac{b_B(\phi)c}{||B||\sin\theta + b_B(\phi)d\sin\beta c}$$

$$\left(2t\sin\theta + (d\sin\beta - t\sin\theta)(tcm_2(\phi) - c_1(\phi)) - \sin\theta\int_0^{\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|}} um_1(\phi)cdu + \right.$$

$$-\sin\theta\int_{\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|}}^{t} ucm_2(\phi) - c_1(\phi)du = \frac{b_B(\phi)c}{||B||\sin\theta + b_B(\phi)d\sin\beta c}$$

$$\left(2t\sin\theta + (d\sin\beta - t\sin\theta)(tcm_2(\phi) - c_1(\phi)) - \frac{\sin\theta}{2}\left(\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|}\right)^2 m_1(\phi)c - \right.$$

$$-\frac{\sin\theta}{2}\left(\left(\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|}\right)^2 - t^2\right)cm_2(\phi) + c_1(\phi)\left(\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|} - t\right)$$

Case (iii) If $\psi_1 + \pi k \leq \phi \leq \pi(k+1) - \psi_2$ and $0 \leq t \leq t_{max}(\omega)$

$$F_U(t,\omega) = \frac{b_B(\phi)c}{||B||\sin\theta + b_B(\phi)d\sin\beta c}$$

$$\left(2t\sin\theta + (d\sin\beta - t\sin\theta)tcm_3(\phi) - \frac{t^2c\sin\theta m_3(\phi)}{2}\right)$$

Case (iv) If $\pi(k+1) - \psi_2 \leq \phi \leq \pi(k+1)$ and $t \leq \frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|}$

$$F_{D_t}(t,\omega) = \frac{b_B(\phi)c}{||B||\sin\theta + b_B(\phi)d\sin\beta c}$$

$$\left(2t\sin\theta + (d\sin\beta - t\sin\theta)tcm_4(\phi) - \frac{t^2c\sin\theta m_4(\phi)}{2}\right)$$

(v) If $\pi k \leq \phi \leq \psi_1 + \pi k$ and $\frac{x_0(\phi)\sin\beta}{||\sin(\beta-\theta)||} \leq t \leq t_{max}(\omega)$

For this case we have 2 sub-cases for calculating $F_T(uc,\phi)$

$$F_T(uc,\phi) = \begin{cases} um_4(\phi)c, & \text{if } u < \frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|} \\ um_5(\phi)c + c_2(\phi), & \text{if } \frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|} \leq u \leq \chi_{max}(\omega) \end{cases}$$

Therefore we get

$$F_{D_t}(t,\omega) = \frac{b_B(\phi)c}{||B||\sin\theta + b_B(\phi)d\sin\beta c}$$

$$\left(2t\sin\theta + (d\sin\beta - t\sin\theta)(tcm_5(\phi) - c_2(\phi)) - \sin\theta\int_0^{\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|}} um_4(\phi)cdu + \right.$$

$$-\sin\theta\int_{\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|}}^{t} ucm_5(\phi) - c_2(\phi)du = \frac{b_B(\phi)c}{||B||\sin\theta + b_B(\phi)d\sin\beta c}$$

$$\left(2t\sin\theta + (d\sin\beta - t\sin\theta)(tcm_5(\phi) - c_2(\phi)) - \frac{\sin\theta}{2}\left(\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|}\right)^2 m_4(\phi)c - \right.$$

$$-\frac{\sin\theta}{2}\left(\left(\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|}\right)^2 - t^2\right)cm_5(\phi) + c_2(\phi)\left(\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|} - t\right)$$

where $\chi_{max}(\phi)$ is defined by (2.6) or (2.7).

If we take $\beta = \pi/2$ then we have same results as in [14].

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