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ON THE SOLITARY SOLUTIONS FOR THE NONLINEAR KLEIN-GORDON EQUATION COUPLED WITH BORN-INFELD THEORY

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Abstract. The aim of this paper is to prove the existence of the nonlinear Klein-Gordon equations coupled with Born-Infeld theory by using variational methods.

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1. Introduction

In recent years, the Born-Infeld nonlinear electromagnetism has become more and more attractive and regained its importance due to its relevance in the theory of superstring and membranes. Mathematically, some people considered the system coupled Klein-Gordon equation with Born-Infeld theory through using variational methods. Furthermore, by variational methods, the existence of solitary wave solution has been studied in different systems, see References [1, 2, 5, 15, 17].

The Born-Infeld (BI) electromagnetic theory [12] was originally proposed as a nonlinear correction of the Maxwell theory in order to overcome the problem of infiniteness in the classical electrodynamics of point particles. The Born–Infeld geometric theory of electromagnetism is a nonlinear generalization of the classical Maxwell theory. The underlying idea was to simply modify the classical theory not to have physical quantities of infinities, that is the principle of finiteness. It was to replace the original Lagrangian density for the Maxwell electrodynamics with a square root form with a parameter b, by which the finiteness of electric fields is ensured.

This paper can be deduced by the search for solutions of the following nonlinear Klein–Gordon equation:

$$\psi_{tt} - \Delta\psi + m^2\psi - |\psi|^{q-2}\psi = 0 \tag{1.1}$$

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with Born-Infeld theory [12]

$$\Sigma_{BI} = \frac{b^2}{4\pi}\left(1 - \sqrt{1 - \frac{1}{b^2}\left(|E|^2 - |B|^2\right)}\right), \tag{1.2}$$

where $\psi = \psi(x,t) \in C, x \in \mathbb{R}^3, t \in \mathbb{R}$, m is a real constant, $b \gg 1$ is the so-called Born–Infeld parameter. It is well known that the classical theory has two difficulties arising from the divergence of energy (see the first section of [11]). Born and Infeld suggested a way to overcome such difficulties, thus introduced the Lagrangian density. Moreover equation (1.2) can be used to develop the theory of electrically charged fields [10]. In addition, E is the the electric field and B is the magnetic induction field. The electromagnetic field is described by the gauge potential (ϕ, A) :

$$\phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}, \qquad A : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3,$$

from (ϕ, A), we get the electric field

$$E = -\nabla\phi - A_t$$

and the magnetic induction field $B = \nabla \times A$.

Suppose that ψ is a charged field and let e denote the eletric charge. The interaction of ψ with the electro-magnetic field is described by the minimal coupling rule, that is, the formal substitution

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + ie\phi, \nabla \to \nabla - ieA \tag{1.3}$$

into the Lagrangian density relative equation (1.1) given by

$$\Sigma_0 = \frac{1}{2}\left[|\frac{\partial\psi}{\partial t}|^2 - |\nabla\psi|^2 - m^2|\psi|^2\right] + \frac{1}{q}|\psi|^q, \tag{1.4}$$

where e denotes the electric charge.

Then equation (1.3) becomes

$$\Sigma_0 = \frac{1}{2}\left[|\frac{\partial\psi}{\partial t} + ie\phi\psi|^2 - |\nabla\psi - ieA\psi|^2 - m^2|\psi|^2\right] + \frac{1}{q}|\psi|^q. \tag{1.5}$$

The total action of the system is $\Xi = \iint \left(\Sigma_{BI} + \Sigma_0\right) \mathrm{d}x\mathrm{d}t$.

In [11], the authors considered the second order expansion of equation (1.2) for

$$\beta = \frac{1}{2b^2} \to 0^+,$$

then they got

$$\Sigma_{BI}^{'} = \frac{1}{4\pi}\left[\frac{1}{2}\left(|E|^2 - |B|^2\right) + \frac{\beta}{4}\left(|E|^2 - |B|^2\right)^2\right],$$

the total action given by $\Xi = \iint \left(\Sigma_{BI}^{'} + \Sigma_0\right) \mathrm{d}x\mathrm{d}t$. Under the electrostatic solitary wave ansatz

$$\psi(x,t) = u(x)e^{i\omega t}, \phi = \phi(x), A = 0,$$

and $e = 1$, where u and ϕ are real valued functions defined on $\mathbb{R}^3$ and ω is a positive frequency parameter, so now

$$\Sigma_{BI}^{'} = \frac{1}{4\pi}\left[\frac{1}{2}\left(|E|^2 - |B|^2\right) + \frac{\beta}{4}\left(|E|^2 - |B|^2\right)^2\right] = \frac{1}{8\pi}|\nabla\phi|^2 + \frac{\beta}{16\pi}|\nabla\phi|^4,$$

therefore the Euler–Lagrange equations associated with the total action Ξ take the the following form

$$\begin{cases} -\Delta u + \left[m^2 - (\omega+\phi)^2\right] u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.6}$$

this type of equations has been found via modern variational methods under various hypotheses on the nonlinear term, see [7, 8, 9, 13, 15]. In [9] the authors found the existence of infinitely many radially symmetric solutions for this problem when $4 < p < 6$ and $|m| > \omega$, in [13] the range $p \in (2, 4]$ was also covered provided $\sqrt{\left(\frac{p}{2} - 1\right)}|m| > \omega$.

Then Chen and Li [7] got the existence of multiple solutions for problem

$$\begin{cases} -\Delta u + \left[m^2 - (\omega+\phi)^2\right] u = |u|^{p-2}u + h(x), & x \in \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.7}$$

when $4 < p < 6$ and $|m| > \omega$ or $2 < p \leqslant 4$ and $\sqrt{\left(\frac{p}{2} - 1\right)}|m| > \omega$.

Later Teng and Zhang [15] got that problem

$$\begin{cases} -\Delta u + \left[m^2 - (\omega+\phi)^2\right] u = |u|^{p-2}u + |u|^{2^*-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.8}$$

has at least a nontrivial solution when $4 < p < 6$ and $m > \omega$ under the electrostatic solitary wave ansatz by using variational methods.

On the other hand, by shrinking the area in problem (1.6), Teng [14] showed some existence and multiple results for the following nonlinear Klein-Gordon equation coupled with Born-Infeld theory in a bounded domain with smooth boundary

$$\begin{cases} -\Delta u + \left[m^2 - (\omega+\phi)^2\right] u = f(x, u), & \text{in } \Omega, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)u^2, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where $m^2 > \frac{\mu}{\mu-2}\omega^2 - \lambda_1$ and f satisfies the following conditions:

(f_1) $f \in C\left(\overline{\Omega} \times \mathbb{R}\right)$ and $f(x, 0) = 0$,

(f_2) There are constants $a_1, a_2 > 0$ such that $|f(x,t)| \leqslant a_1 + a_2|t|^s$, where $1 < s < \frac{n+2}{n-2}(n \geqslant 3)$,

(f_3) $\lim\limits_{t\to 0} \frac{f(x,t)}{t} = 0$,

(f_4) There exists $\mu > 2$ and $\mathbb{R} \geqslant 0$ such that $tf(x,t) \geqslant \mu F(x,t) > 0$ for $|t| \geqslant \mathbb{R}$ and $x \in \Omega$,

or $m^2 > \frac{\mu}{\mu-2}\omega^2 - \lambda_1$ and f satisfies the conditions above and an extra condition:

(f_5) $f(x,-u) = -f(x,u)$ for all $u \in \mathbb{R}$ and $x \in \Omega$.

In addition, the authors in [1] proved the existence of nontrivial ground state solution for the following nonlinear Klein–Gordon equation coupled with Born–Infeld theory in $\mathbb{R}^2$ involving unbounded or decaying radial potentials

$$(1.10) \qquad \begin{cases} -\Delta u + \left[m^2 - (\omega+\phi)^2\right] V(|x|)u = K(|x|)f(u), & \text{in } \mathbb{R}^2, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi\,(\omega+\phi)\,V(|x|)u^2, & \text{in } \mathbb{R}^2, \end{cases}$$

where $V, K : \mathbb{R}^2 \to \mathbb{R}$ are radial potentials which may be unbounded, singular at the origin or vanishing at infinity and the nonlinear term $f(s)$ is allowed to enjoy a critical exponential growth.

Recently, Che and Chen in [6] proved the existence of infinitely many negative-energy solutions for the following system via the genus properties in critical point theory

$$(1.11) \qquad \begin{cases} -\Delta u + V(x)u - (2\omega+\phi)\,\phi u = f\,(x,u)\,, & x \in \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi\,(\omega+\phi)\,u^2, & x \in \mathbb{R}^3, \end{cases}$$

where the functions $V(x)$ and $f(x,u)$ satisfy the following hypotheses.

(V_1) $V \in C\left(\mathbb{R}^3\right)$ satisfies $\inf_{x\in\mathbb{R}^3} V(x) \geqslant a > 0$, where $a > 0$ is a constant. Moreover, for any $M > 0, meas\{x \in \mathbb{R}^3 : V(x) \leqslant M\} < \infty$, where $meas$ denotes the Lebesgue measure in $\mathbb{R}^3$.

(1) $f \in C\left(\mathbb{R}^3 \times \mathbb{R}\right)$ and there exists $1 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < 2, m \in \mathbb{N}, m \geqslant 1, c_i(x) \in L^{\frac{2}{2-\alpha_i}}\left(\mathbb{R}^3, \mathbb{R}^+\right)$ such that

$$|f(x,u)| \leqslant \sum_{i=1}^{m} \alpha_i c_i(x)|u|^{\alpha_i-1}, \qquad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

(2) There exists a bounded open set $J \subset \mathbb{R}^3$ and three constants $a_1, a_2 > 0$ and $a_3 \in (1,2)$ such that

$$F(x,u) \geqslant a_2|u|^{a_3}, \qquad \forall (x,u) \in J \times [-a_1, a_1]\,,$$

where $F(x,u) = \int_0^u f(x,s)ds$.

(3) $f(x,u) = -f(x,-u)$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$.

Immediately after the previous equation, Wen, Tang and Chen in [16] proved the existence of infinitely many solutions and least energy solutions for the nonhomogeneous Klein-Gordon equation coupled with Born-Infeld theory.

For general potential $a(x)$ and the nonlinearity $f(x,u) = \lambda K(x)|u|^{q-2}u+g(x)|u|^{p-2}u$, Chen and Song in [8] studied this system

(1.12)
$$\begin{cases} -\Delta u + a(x)u - (2\omega+\phi)\,\phi u = \lambda K(x)|u|^{q-2}u + g(x)|u|^{p-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi\,(\omega+\phi)\,u^2, & x \in \mathbb{R}^3, \end{cases}$$

and proved the existence of multiple solutions for Klein–Gordon equation with concave and convex nonlinearities coupled with Born–Infeld theory when a, k, g are measurable functions satisfying the following conditions:

(a_1) $a(x) \in C(\mathbb{R}^3)$ satisfying $a_0 := \inf\limits_{x\in\mathbb{R}^3} a(x) > 0$.

(k) $k(x) \in L^{\frac{12q}{(6-q)(1+q)}}\left(\mathbb{R}^3\right), k(x) \geqslant 0$ for $a.e.$ $x \in \mathbb{R}^3$ and $k(x) \neq 0$.

(g) $g(x) \in L^{\infty}\left(\mathbb{R}^3\right), g(x) \geqslant 0$ for $a.e.$ $x \in \mathbb{R}^3$ and $g(x) \neq 0$.

The main idea of this paper is to establish the existence of solitary wave solutions of the following Klein-Gordon equation coupled with Born-Infeld theory:

$$\begin{cases} -\Delta u + \eta(x)u - (2\omega+\phi)\,\phi u = \mu K(x)|u|^{q-2}u + |u|^{2^*-2}u, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi\,(\omega+\phi)\,u^2, \end{cases} \tag{1.13}$$

where ω and μ are positive constants, $\beta \gg 1$ is a constant, $\eta(x) \in C(\mathbb{R}^3), K(x) \in L^{\infty}(\mathbb{R}^3), 4 \leqslant q < 2^* = \frac{2N}{N-2}$. Since we define in three-dimensional space in this paper, after that $2^* = 6$.

In this case, the functional F corresponding to problem (1.13) defined by

$$\begin{aligned} F(u,\phi) = \int \Bigg[&\frac{1}{2}|\nabla u|^2 + \frac{1}{2}\eta(x)u^2 - \frac{1}{2}(2\omega+\phi)\phi u^2 \\ &- \frac{1}{8\pi}\left|\nabla\phi\right|^2 - \frac{\beta}{16\pi}\left|\nabla\phi\right|^4 - \frac{\mu}{q}K(x)|u|^q - \frac{1}{6}|u|^6 \Bigg], \end{aligned} \tag{1.14}$$

which by a standard argument is C^1 on $H(\mathbb{R}^3) \times D(\mathbb{R}^3)$, the definitions of $H(\mathbb{R}^3)$ and $D(\mathbb{R}^3)$ will be given later. Here and hereafter, $\int \cdot$ denotes $\int_{\mathbb{R}^3} \cdot \, \mathrm{d}x$.

Remark 1.1. The functional F is strongly indefinite, i.e. unbounded from below and from above on infinite subspaces. In order to avoid this indefiniteness, we can borrow the reduction methods.

2. Main results

Firstly, assume that the system (1.13) satisfies the following conditions:

(i) $\eta(x) \geqslant 0$ is a radial function, that is, $\eta(x) = \eta(r), r = |x|$,

(ii) $K : \mathbb{R}^3 \to \mathbb{R}$ is a radial function, moreover $0 \leqslant K(x) < \Lambda$ and $K(x) \not\equiv 0$ for $a.e.$ $x \in \mathbb{R}^3$, where $\Lambda > 0$ is a constant.

Next some notations are given. For all $1 \leqslant s \leqslant +\infty$, $L^s(\mathbb{R}^3)$ denotes a Lebesgue space with the norm given by $|\cdot|_{L^s}$.

Let $D^{1,2}(\mathbb{R}^3)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ endowed with the norm

$$\|u\|_{D^{1,2}}^2 = \int |\nabla u|^2.$$

The space $H^1(\mathbb{R}^3)$ is endowed with the norm

$$\|u\|_{H^1}^2 = \int \left(|\nabla u|^2 + u^2\right).$$

$D(\mathbb{R}^3)$ denotes the the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_D = \left(\int |\nabla u|^2\right)^{\frac{1}{2}} + \left(\int |\nabla u|^4\right)^{\frac{1}{4}}.$$

Define

$$H = \{u \in H^1(\mathbb{R}^3) : \int \left[|\nabla u|^2 + \eta(x)u^2\right] < \infty\}$$

is a Hilbert space, whose inner product and norm are given, respectively

$$(u, v) = \int \left(\nabla u \cdot \nabla v + \eta(x)uv\right), \quad \|u\|^2 = (u, u).$$

Obviously, by the Poincaré inequality, the embedding $H(\mathbb{R}^3) \hookrightarrow H^1(\mathbb{R}^3)$ is continuous and $D(\mathbb{R}^3)$ is continuously embedded in $D^{1,2}(\mathbb{R}^3)$. Moreover, from Sobolev's imbedding theorem (see [11]), $D(\mathbb{R}^3)$ is continuously embedded in$L^\infty(\mathbb{R}^3)$.

In this paper, we show the following results about the system (1.13):

Theorem 2.1. *Suppose (i)-(ii) hold, if $4 < q < 6$, then for each $\mu > 0$ the problem* (1.13) *admits a radially symmetric solution.*

Theorem 2.2. *Suppose (i)-(ii) hold, if $q = 4$, then for sufficiently large $\mu > 0$, the problem* (1.13) *still possesses a radially symmetric solution.*

Moreover, we have the following lemma about the second equation of problem (1.13).

Lemma 2.1. (*a*) *For any $u \in H(\mathbb{R}^3)$, there exists a unique $\Phi(u) = \phi \in D(\mathbb{R}^3)$ which satisfies $\Delta\Phi(u) + \beta\Delta_4\Phi(u) = 4\pi\left(\omega + \Phi(u)\right)u^2$,*
(*b*) *If u is radially symmetric, then $\Phi(u)$ is also radially symmetric,*
(*c*) *For any $u \in H(\mathbb{R}^3)$, it results in $\Phi(u) \leqslant 0$. Moreover, $\Phi(u)(x) \geqslant -\omega$, provided $u(x) \neq 0$.*

The results were proved by Lemma 3 in [9], Lemma 5 in [9], Lemma 2.3 in [13], respectively. Similar to the Proposition 1.1 in Reference [5], we have the following lemma.

Lemma 2.2. *The map ϕ is C^1 and $G_\phi = \{(u, \phi) \in H(\mathbb{R}^3) \times D(\mathbb{R}^3) \big| F'_\phi(u, \phi) = 0\}$.*

Proof. Since

$$F(u,\Phi(u)) = \int \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}\eta(x)u^2 - \frac{1}{2}\left(2\omega + \Phi(u)\right)\Phi(u)u^2 - \frac{1}{8\pi}\left|\nabla\Phi(u)\right|^2\right] + \int \left[-\frac{\beta}{16\pi}\left|\nabla\Phi(u)\right|^4 - \frac{\mu}{q}K(x)|u|^q - \frac{1}{6}|u|^6\right],$$

then

$$F'_\phi\left(u,\Phi(u)\right) = \int \left[-\frac{1}{4\pi}\left|\nabla\Phi(u)\right|^2 - \frac{\beta}{4\pi}\left|\nabla\Phi(u)\right|^4 - \omega\Phi(u)u^2 - \Phi^2(u)u^2\right]. \tag{2.1}$$

On the other hand, from the second equation in problem (1.13), one gets

$$-\int \left(\left|\nabla\Phi(u)\right|^2 + \beta\left|\nabla\Phi(u)\right|^4\right) = \int 4\pi\left(\omega + \Phi(u)\right)\Phi(u)u^2,$$

i.e.,

$$\int \left[\frac{1}{4\pi}\left|\nabla\Phi(u)\right|^2 + \frac{\beta}{4\pi}\left|\nabla\Phi(u)\right|^4\right] = \int\left[-\omega\Phi(u)u^2 - \Phi^2(u)u^2\right]. \tag{2.2}$$

Therefore, according to equation (2.1), $F'_\phi\left(u,\phi\right) = 0$. So now we define $I(u) = F(u,\phi)$ in $H(\mathbb{R}^3)$. □

By Lemma 2.2, we have

$$I'(u) = F'_u\left(u,\Phi(u)\right) + F'_\phi\left(u,\Phi(u)\right)\Phi'(u) = F'_u\left(u,\Phi(u)\right),$$

and if $u, v \in H(\mathbb{R}^3)$, one gets

$$I'(u)v = \int \left[\nabla u\cdot\nabla v + \eta(x)uv - (2\omega+\phi)\phi uv - \mu K(x)|u|^{q-2}uv - |u|^4uv\right]. \tag{2.3}$$

Lemma 2.3. *The following statements are equivalent:*
(a) $(u,\phi) \in H(\mathbb{R}^3)\times D(\mathbb{R}^3)$ *is a solution of system* (1.13)*,*
(b) u *is a critical point for* I *and* $\phi = \Phi(u)$*.*

Proof. $(b) \Longrightarrow (a)$ Obviously.
$(a) \Longrightarrow (b)$ Suppose $F'_u\left(u,\phi\right)$ and $F'_\phi\left(u,\phi\right)$ denote the partial derivatives of F at $(u,\phi) \in H(\mathbb{R}^3)\times D(\mathbb{R}^3)$. Then for every $v \in H(\mathbb{R}^3)$ and $\psi \in D(\mathbb{R}^3)$, one gets

$$F'_u\left(u,\phi\right)[v] = \int \left[\nabla u\cdot\nabla v + \eta(x)uv - (2\omega+\phi)\phi uv - \mu K(x)|u|^{q-2}uv - |u|^4uv\right], \tag{2.4}$$

$$F'_\phi\left(u,\phi\right)[\psi] = \int \left[-\frac{1}{2\pi}\nabla\phi\nabla\psi - \frac{\beta}{\pi}\left|\nabla\phi\right|^2\phi\psi - \omega\psi u^2 - 2\phi\psi u^2\right]. \tag{2.5}$$

By the standard computations, we can prove that $F'_u\left(u,\phi\right)$ and $F'_\phi\left(u,\phi\right)$ are continuous. From equations (2.4) and (2.5), it is easy to obtain that its critical points are solutions of problem (1.13), by (a) of Lemma 2.1, one has $\phi = \Phi(u)$. □

Due to the presence of the critical growth, the Sobolev embedding $H(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)(2 \leqslant p \leqslant 6)$ is not compact and then it is usually difficult to prove that a Palais–Smale sequence is strongly convergent when we seek solutions of problem

(1.13) by variational methods. A standard tool to overcome the problem is to restrict ourselves to radial functions, namely we look at the functional I on the subspace $H_r(\mathbb{R}^3) = \{u \in H(\mathbb{R}^3) \big| u(x) = u(|x|)\}$, compactly embedded in $L_r^p(\mathbb{R}^3)$ for $2 < p < 6$. Moreover, from [2], for all $u \in H(\mathbb{R}^3)$, for any $g \in O(3)$, we have

$$I(T_g u) = I(u).$$

By standard arguments, one sees that if a critical point $u \in H_r(\mathbb{R}^3)$ for the functional $I\big|_{H_r(\mathbb{R}^3)}$ is also a critical point of I.

3. The Proof of Theorem 2.1

Firstly, we prove the functional I possesses the Mountain-Pass geometry. From the second equation of system (1.13), one obtains equation (2.2), combining equation (1.14) with (2.2), one gets

$$\begin{aligned}
I(u) = F(u,\phi) &= \int \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}\eta(x)u^2 - \frac{1}{2}(2\omega+\phi)\phi u^2\right] \\
&\quad + \int \left[-\frac{1}{8\pi}|\nabla\phi|^2 - \frac{\beta}{16\pi}|\nabla\phi|^4 - \frac{\mu}{q}K(x)|u|^q - \frac{1}{6}|u|^6\right] \\
&= \int \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}\eta(x)u^2 + \frac{1}{8\pi}|\nabla\phi|^2 + \frac{3\beta}{16\pi}|\nabla\phi|^4 + \frac{1}{2}\phi^2 u^2\right] \\
&\quad - \int \left[\frac{\mu}{q}K(x)|u|^q + \frac{1}{6}|u|^6\right].
\end{aligned} \tag{3.1}$$

By the Sobolev inequality, one has $I(u) \geqslant C_1\|u\|^2 - C_2\|u\|^q - C_3\|u\|^6$, then there exists $\alpha, \rho > 0$ such that $\inf\limits_{\|u\|=\rho} I(u) > \alpha$. In addition, from equation (1.14), there exists a function $u \in H_r(\mathbb{R}^3) \setminus \{0\}$, it is easy to obtain

$$\begin{aligned}
\lim_{t\to+\infty} I(tu) &= \int \left[\frac{t^2}{2}|\nabla u|^2 + \frac{t^2}{2}\eta(x)u^2 - \frac{t^2}{2}(2\omega+\Phi(tu))\Phi(tu)u^2 - \frac{1}{8\pi}|\nabla\Phi(tu)|^2\right] \\
&\quad + \int \left[-\frac{\beta}{16\pi}|\nabla\Phi(tu)|^4 - \frac{\mu t^q}{q}K(x)|u|^q - \frac{t^6}{6}|u|^6\right] \\
&\leqslant \frac{t^2}{2}\int \left[|\nabla u|^2 + \eta(x)u^2 - 2\omega\Phi(tu)u^2 - \frac{2\mu t^{q-2}}{q}K(x)|u|^q - \frac{t^4}{3}|u|^6\right] \\
&\leqslant -\infty,
\end{aligned}$$

which implies that $I(u) \to -\infty$, as $\|u\| \to \infty$. In particular, there exists $u_1 \in H_r(\mathbb{R}^3)$, $\|u_1\| > \rho$ such that $I(u_1) < 0$. Define

$$c = \inf_{\gamma\in\Gamma}\max_{0\leqslant t\leqslant 1} I(\gamma(t)), \tag{3.2}$$

where $\Gamma = \{\gamma \in C\left([0,1], H_r(\mathbb{R}^3)\right) \big| \gamma(0) = 0, \gamma(1) = u_1\}$ is the MP level. Obviously, $c \geqslant \alpha > 0$. There exists a $(PS)_c$ sequence $\{u_n\} \subset E$ such that

$$\begin{aligned}
&I(u_n) \to c, \\
&I'(u_n) \to 0, \quad n \to \infty.
\end{aligned} \tag{3.3}$$

Lemma 3.1. *The $(PS)_c$ sequence $\{u_n\} \subset E$ given in equation (3.3) is bounded.*

Proof. There is a positive constant M such that

$$
\begin{aligned}
&M + o(1)\|u_n\| \geqslant I(u_n) - \frac{1}{q}(I'(u_n), u_n) \\
&= \left(\frac{1}{2} - \frac{1}{q}\right)\int \left[|\nabla u_n|^2 + \eta(x)u_n^2\right] + \frac{1}{8\pi}\int |\nabla\Phi(u_n)|^2 + \frac{3\beta}{16\pi}\int |\nabla\Phi(u_n)|^4 \\
&\quad + \left(\frac{1}{2} + \frac{1}{q}\right)\int \Phi^2(u_n)u_n^2 + \left(\frac{1}{q} - \frac{1}{6}\right)\int |u_n|^6 + \frac{2}{q}\int \omega\Phi(u_n)u_n^2.
\end{aligned} \tag{3.4}
$$

Substituting equation (2.2) into equation (3.4), we get

$$
\begin{aligned}
M + o(1)\|u_n\| &\geqslant I(u_n) - \frac{1}{q}(I'(u_n), u_n) \\
&= \left(\frac{1}{2} - \frac{1}{q}\right)\int \left[|\nabla u_n|^2 + \eta(x)u_n^2\right] + \left(\frac{1}{8\pi} - \frac{1}{2q\pi}\right)\int |\nabla\Phi(u_n)|^2 \\
&\quad + \left(\frac{1}{2} - \frac{1}{q}\right)\int \Phi^2(u_n)u_n^2 \\
&\quad + \left(\frac{3\beta}{16\pi} - \frac{\beta}{2q\pi}\right)\int |\nabla\Phi(u_n)|^4 + \left(\frac{1}{q} - \frac{1}{6}\right)\int |u_n|^6 \geqslant C_4\|u_n\|^2.
\end{aligned}
$$

Since $4 < q < 6$, as a consequence of the above inequality, $\{u_n\}$ is bounded in $H_r(\mathbb{R}^3)$. $\square$

Furthermore, by equation (2.2), one gets

$$
\int \left(|\nabla\Phi(u_n)|^2 + \beta\,|\nabla\Phi(u_n)|^4\right) = -4\pi\int \omega\Phi(u_n)u_n^2 - 4\pi\int \Phi^2(u_n)u_n^2. \tag{3.5}
$$

Then by Hölder inequality and Sobolev inequality, one obtains

$$
\int \left(|\nabla\Phi(u_n)|^2 + \beta\,|\nabla\Phi(u_n)|^4\right) \leqslant C_5\|\Phi(u_n)\|_{D_r}\|u_n\|_{H_r}^2.
$$

So $\{\Phi(u_n)\}$ is bounded (even uniformly). Up to subsequence, we may assume that there exists $u \in H_r(\mathbb{R}^3)$ and $\varphi \in D_r(\mathbb{R}^3)$ such that

$$
u_n \rightharpoonup u \quad \text{in } H_r(\mathbb{R}^3), \tag{3.6}
$$

$$
u_n \to u \quad \text{in } L_r^s(\mathbb{R}^3) \text{ for } 2 < s < 6, \tag{3.7}
$$

$$
\Phi(u_n) \rightharpoonup \varphi \quad \text{in } D_r(\mathbb{R}^3). \tag{3.8}
$$

Lemma 3.2. *$\varphi = \Phi(u)$ and $\Phi(u_n) \to \Phi(u)$ in $D_r(\mathbb{R}^3)$.*

Proof. First we prove the uniqueness. For every fixed $u \in H_r(\mathbb{R}^3)$, we consider the following minimizing problem $\inf\limits_{\phi\in D_r} E_u(\phi)$, where $E_u : D_r \to \mathbb{R}$ defined as energy functional of the second equation in system (1.13).

$$
E_u(\phi) = \int \left[\frac{1}{8\pi}\,|\nabla\phi|^2 + \frac{\beta}{16\pi}\,|\nabla\phi|^4 + \omega\phi u^2 + \frac{1}{2}\phi^2u^2\right].
$$

In fact, by the proof of Lemma 2.1 in [17], one can know

$$\Phi(u_n) \to \varphi, \text{ locally uniformly in } \mathbb{R}^3,$$

so we obtain

$$\int \Phi(u_n)u_n^2 \to \int \varphi u^2, \qquad \int \Phi^2(u_n)u_n^2 \to \int \varphi^2 u^2.$$

From the weak lower semicontinuity of the norm in D_r and the convergence above, one has

$$E_u(\varphi) \leqslant \liminf_{n\to\infty} E_{u_n}(\Phi(u_n)) \leqslant \liminf_{n\to\infty} E_{u_n}(\Phi(u)) = E_u(\Phi(u)),$$

then by (a) of Lemma 2.1, $\varphi = \Phi(u)$.

Next we prove that $\{\Phi(u_n)\}$ converges strongly in D_r. Since $\Phi(u_n)$ and $\Phi(u)$ satisfy the second equation in problem (1.13).

$$\begin{cases} \int \Big[\nabla\Phi(u_n)\cdot\nabla v + \beta\left|\nabla\Phi(u_n)\right|^3\cdot\nabla v\Big] = \int \left[-4\pi\omega u_n^2 v - 4\pi\Phi(u_n)u_n^2 v\right], \\ \int \Big[\nabla\Phi(u)\cdot\nabla v + \beta\left|\nabla\Phi(u)\right|^3\cdot\nabla v\Big] = \int \left[-4\pi\omega u^2 v - 4\pi\Phi(u)u^2 v\right], \end{cases}$$

then we take the difference for Φ to have

(3.9)

$$\begin{aligned} &\int \Big[\nabla\left(\Phi(u_n)-\Phi(u)\right)\cdot\nabla v + \beta\Big(\left|\nabla\Phi(u_n)\right|^2\nabla\Phi(u_n) - \left|\nabla\Phi(u)\right|^2\nabla\Phi(u)\Big)\cdot\nabla v\Big] \\ &= -4\pi\int \left[\omega\left(u_n^2-u^2\right)v + \left(\Phi(u_n)u_n^2 - \Phi(u)u^2\right)v\right], \qquad\qquad v\in D_r(\mathbb{R}^3). \end{aligned}$$

Testing with $v = (\Phi(u_n)-\Phi(u))$ the following holds:

$$\begin{aligned} &C_6\Big(\left|\nabla\Phi(u_n)-\nabla\Phi(u)\right|_{L_r^2}^2 + \left|\nabla\Phi(u_n)-\nabla\Phi(u)\right|_{L_r^4}^4\Big) \\ &\leqslant -4\pi\int\left[w(u_n^2-u^2)v + \left(\Phi(u_n)u_n^2-\Phi(u)u^2\right)v\right] \\ &= -4\pi\int\left[w(u_n^2-u^2)v + u_n^2\left(\Phi(u_n)-\Phi(u)\right)v + (u_n^2-u^2)\Phi(u)v\right], \end{aligned}$$

the above equation holds since we have inequality

$$\left[\left(|x|^{p-2}x-|y|^{p-2}y\right)(x-y)\right] \geqslant C_p|x-y|^p, \qquad x,y\in\mathbb{R}^N, p\geqslant 2.$$

By Hölder inequality and Sobolev inequality, one has

$$\begin{aligned} &\left|\nabla\Phi(u_n)-\nabla\Phi(u)\right|_{L_r^2}^2 + \left|\nabla\Phi(u_n)-\nabla\Phi(u)\right|_{L_r^4}^4 \\ &\leqslant |4\pi\omega|\int\left[\left|u_n^2-u^2\right|\left|\Phi(u_n)-\Phi(u)\right|\right] + 4\pi\int\left[\left|u_n^2-u^2\right|\left|\Phi(u)\right|\left|\Phi(u_n)-\Phi(u)\right|\right] \\ &\leqslant |4\pi\omega|\left|\Phi(u_n)-\Phi(u)\right|_{L_r^6}\left|u_n^2-u^2\right|_{L_r^{\frac{6}{5}}} \\ &\quad + 4\pi\left|\Phi(u)\right|_{L_r^6}\left|\Phi(u_n)-\Phi(u)\right|_{L_r^6}\left|u_n^2-u^2\right|_{L_r^{\frac{3}{2}}} \leqslant C_7\left|u_n-u\right|_{L_r^{\frac{12}{5}}} + C_8\left|u_n-u\right|_{L_r^3}. \end{aligned}$$

Thus $\Phi(u_n)\to\Phi(u)$ strongly in $D_r(\mathbb{R}^3)$. $\square$

Lemma 3.3. *The weak limit* $(u, \Phi(u))$ *solves problem* (1.13).

Proof.

$$(I'(u_n), v) = \int \left[\nabla u_n \cdot \nabla v + \eta(x) u_n v - (2\omega + \Phi(u_n))\, \Phi(u_n) u_n v\right] - \int \left[\mu K(x) |u_n|^{q-2} u_n v + |u_n|^4 u_n v\right]. \tag{3.10}$$

All convergences in the sequel must be understood passing to a subsequence if necessary. Since $\{u_n\}$ is bounded in $L^6_r(\mathbb{R}^3)$, it follows

$$|u_n|^4 u_n \rightharpoonup |u|^4 u, \qquad \text{in } (L^6_r(\mathbb{R}^3))^*.$$

Moreover by Lemma 3.2, one gets

$$\int u_n \Phi^2(u_n) v + 2\omega \int \Phi(u_n) u_n v \to \int u \Phi^2(u) v + 2\omega \int \Phi(u) u v, \qquad v \in H_r(\mathbb{R}^3).$$

In fact one obtains

$$\int |\Phi(u)u - \Phi(u_n)u_n|\, |v| \leqslant |\Phi(u) - \Phi(u_n)|_{L^6_r} |u|_{L^3_r} |v|_{L^2_r} + |\Phi(u_n)|_{L^6_r} |v|_{L^2_r} |u_n - u|_{L^3_r} \tag{3.11}$$

and

$$\int |u_n \Phi^2(u_n) - u\Phi^2(u)||v| \leqslant |u_n - u|_{L^3_r} |\Phi(u_n)|^2_{L^6_r} |v|_{L^3_r} + |\Phi(u_n) - \Phi(u)|_{L^6_r} |\Phi(u_n) + \Phi(u)|_{L^6_r} |u|_{L^6_r} |v|_{L^2_r}. \tag{3.12}$$

The compactness of the embedding $H_r(\mathbb{R}^3) \hookrightarrow L^q_r(\mathbb{R}^3)$ the lemma follows. $\square$

Due to the lack of compactness, which prevents us to prove that u_n converges strongly in $H_r(\mathbb{R}^3)$, we do not know yet whether $u \neq 0$. In order to overcome this difficulty, we need let c denote the MP level.

Lemma 3.4. *Since functions are defined in dimension* $N = 3$, *then we can get* $c < \frac{1}{3} S^{\frac{3}{2}}$, *where* S *corresponds to the best constant for the Sobolev embedding* $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, *precisely,*

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int |\nabla u|^2}{\left(\int |u|^6\right)^{\frac{1}{3}}}.$$

Proof. Now given ε, we consider the Talenti function [3] $u_\varepsilon \in D^{1,2}(\mathbb{R}^3)$ defined by

$$u_\varepsilon = C \frac{\varepsilon^{\frac{1}{4}}}{(\varepsilon + |x|^2)^{\frac{1}{2}}},$$

where $C > 0$ is a normalized constant. Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that $0 \leqslant \varphi \leqslant 1$, and there exists $R > 0$ such that $\varphi\big|_{B_R} \equiv 1$, $supp\varphi \subset B_{2R}$. Set $W_\varepsilon = \varphi u_\varepsilon$ and define $V_\varepsilon := \frac{W_\varepsilon}{|W_\varepsilon|_{L^6_r}}$. From the estimates obtained in [4] we get, as $\varepsilon \to 0$,

$$X_\varepsilon := |\nabla V_\varepsilon|^2_{L^2_r} \leqslant S + O\left(\varepsilon^{\frac{1}{2}}\right), \tag{3.13}$$

$$|V_\varepsilon|^2_{L^2_r} = O\left(\varepsilon^{\frac{1}{2}}\right). \tag{3.14}$$

Since as $t \to +\infty, I(tV_\varepsilon) \to -\infty$, we may assume that $\sup\limits_{t\geqslant 0} I(tV_\varepsilon) = I(t_\varepsilon V_\varepsilon)$ and without loss of generality that $t_\varepsilon \geqslant C_0 > 0$, for all $\varepsilon > 0$ (otherwise we could find a sequence $\varepsilon_n \to 0$ such that $t_{\varepsilon_n} V_{\varepsilon_n} \to 0$ contradicting that $c > 0$). Next for any $\varepsilon > 0$ small enough, the following estimate holds

$$t_\varepsilon \leqslant \left(X_\varepsilon + \int \left(\eta(x) + 2\omega^2\right) V_\varepsilon^2\right)^{\frac{1}{4}} = t_0. \tag{3.15}$$

Let $f(t) = I(tV_\varepsilon)$ and compute

$$\begin{aligned}
f'(t) &= (I'(tV_\varepsilon), V_\varepsilon)\\
&= \int \left[t|\nabla V_\varepsilon|^2 + \eta(x)tV_\varepsilon^2 - (2\omega + \Phi(tV_\varepsilon))\,\Phi(tV_\varepsilon)tV_\varepsilon^2 - \mu t^{q-1}K(x)|V_\varepsilon|^q - t^5|V_\varepsilon|^6\right]\\
&\leqslant \int \left[t|\nabla V_\varepsilon|^2 + \eta(x)tV_\varepsilon^2 - 2\omega\Phi(tV_\varepsilon)tV_\varepsilon^2 - t^5|V_\varepsilon|^6\right]\\
&\leqslant \int \left[t|\nabla V_\varepsilon|^2 + \eta(x)tV_\varepsilon^2 + 2\omega^2 tV_\varepsilon^2 - t^5|V_\varepsilon|^6\right]\\
&= t\int \left[|\nabla V_\varepsilon|^2 + \eta(x)V_\varepsilon^2 + 2\omega^2 V_\varepsilon^2\right] - t^5 = tt_0^4 - t^5 \leqslant 0, \qquad t \geqslant t_0.
\end{aligned}$$

Thus equation (3.15) holds true. From the second equation in system (1.13), one gets

$$\int \left(\frac{1}{16\pi}|\nabla\phi|^2 + \frac{\beta}{16\pi}|\nabla\phi|^4\right) = -\frac{1}{4}\int (\omega + \phi)\,\phi u^2. \tag{3.16}$$

Now substituting this equation into the functional $I(u)$, one has

(3.17)

$$I(u) = \int \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}\eta(x)u^2 - \frac{3}{4}\omega\phi u^2 - \frac{1}{4}\phi^2u^2 - \frac{1}{16\pi}|\nabla\phi|^2 - \frac{\mu}{q}K(x)|u|^q - \frac{1}{6}|u|^6\right].$$

In view of equation (3.16), we have

$$-\frac{1}{4}\int \phi^2u^2 \leqslant \int \omega^2u^2 \tag{3.18}$$

and the function $j(t) = \frac{1}{2}t^2t_0^4 - \frac{1}{6}t^6$ is increasing on $[0, t_0)$, then by equations (3.13), (3.17), (3.18) and (c) of Lemma 2.1, one obtains

$$\begin{aligned}
I(t_\varepsilon V_\varepsilon) &= \int \left[\frac{t_\varepsilon^2}{2}\left(|\nabla V_\varepsilon|^2 + \eta(x)V_\varepsilon^2\right) - \frac{t_\varepsilon^2}{4}\Phi^2(t_\varepsilon V_\varepsilon)V_\varepsilon^2 - \frac{1}{16\pi}|\nabla\Phi(t_\varepsilon V_\varepsilon)|^2\right]\\
&\quad + \int \left[-\frac{3t_\varepsilon^2}{4}\omega\Phi(t_\varepsilon V_\varepsilon)V_\varepsilon^2 - \frac{\mu t_\varepsilon^q}{q}K(x)|V_\varepsilon|^q - \frac{t_\varepsilon^6}{6}|V_\varepsilon|^6\right]\\
&\leqslant \int \left[\frac{t_\varepsilon^2}{2}\left(|\nabla V_\varepsilon|^2 + \left(\eta(x) + 2\omega^2\right)V_\varepsilon^2\right)\right] + \int \left[-\frac{3t_\varepsilon^2}{4}\omega\Phi(t_\varepsilon V_\varepsilon)V_\varepsilon^2 - \frac{\mu t_\varepsilon^q}{q}K(x)|V_\varepsilon|^q - \frac{t_\varepsilon^6}{6}|V_\varepsilon|^6\right]\\
&\leqslant \frac{1}{3}\left(S + O\left(\varepsilon^{\frac{1}{2}}\right) + \int \left(\eta(x) + 2\omega^2\right)V_\varepsilon^2\right)^{\frac{3}{2}} + \frac{3t_\varepsilon^2}{4}\omega^2\int V_\varepsilon^2 - \frac{\mu t_\varepsilon^q}{q}\int K(x)|V_\varepsilon|^q,
\end{aligned}$$

then using the inequality $(a+b)^{\delta} = a^{\delta} + \delta(a+b)^{\delta-1} b$, for all $\delta \geqslant 1, a, b \geqslant 0$, we get

$$I(t_\varepsilon V_\varepsilon) \leqslant \frac{1}{3} S^{\frac{3}{2}} + O\left(\varepsilon^{\frac{1}{2}}\right) + C_1(\varepsilon) \int V_\varepsilon^2 - \mu C_2(\varepsilon) \int |V_\varepsilon|^q,$$

with constants $C_i(\varepsilon) \geqslant C_0 > 0\,(i=1,2)$. On the other hand, we may get the conclusion that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{\frac{1}{2}}} \int \left(V_\varepsilon^2 - \mu |V_\varepsilon|^q\right) = -\infty \ \textit{for } \varepsilon \textit{ small enough.} \tag{3.19}$$

In fact, by the definition of W_ε, since for $\varepsilon \to 0$, as in [3],

$$\int_{B_{2R}} |W_\varepsilon|^6 \mathrm{d}x = \int_{B_{2R}} |\varphi u_\varepsilon|^6 \mathrm{d}x = C \int \frac{1}{(1+|x|^2)^3} + O\left(\varepsilon^{\frac{3}{2}}\right). \tag{3.20}$$

It suffices to evaluate (3.19) with W_ε in place of V_ε, one has for $p \geqslant 1$,

(3.21)

$$|u_\varepsilon|_{L_r^p}^p = \int_{B_R} \frac{\varepsilon^{\frac{p}{4}}}{(\varepsilon+|x|^2)^{\frac{p}{2}}} \mathrm{d}x = C \int_0^R \frac{\varepsilon^{-\frac{p}{4}} s^2}{\left(1+\left(\frac{s}{\sqrt{\varepsilon}}\right)^2\right)^{\frac{p}{2}}} \mathrm{d}s = C \varepsilon^{\frac{6-p}{4}} \int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^2}{(1+r^2)^{\frac{p}{2}}} \mathrm{d}r,$$

while

$$\int_{B_{2R} \setminus B_R} |W_\varepsilon|^p \mathrm{d}x = O\left(\varepsilon^{\frac{p}{4}}\right), \qquad \varepsilon \to 0, \tag{3.22}$$

and therefore, one has for $4 < q < 6$, as $\varepsilon \to 0$,

$$\int_{B_{2R}} W_\varepsilon^2 \mathrm{d}x - \mu \int_{B_{2R}} W_\varepsilon^q \mathrm{d}x \leqslant C_9 \varepsilon^{\frac{1}{2}} - C_{10} \mu \varepsilon^{\frac{6-q}{4}}, \tag{3.23}$$

where $C_i > 0\,(i=9,10)$ are independent from ε. According to equations (3.20) and (3.23), we conclude the proof of equation (3.19).

Now we only need prove $u \neq 0$. Assume that the lemma holds true, by contradiction, $u = 0$, (and hence $\Phi(u) = 0$). Since, as $n \to \infty$, $(I'(u_n), u_n) \to 0$, $u_n \to 0$ in $L_r^s(\mathbb{R}^3)$. Obviously, $\int \left[u_n^2 \Phi^2(u_n) + 2\omega \Phi(u_n) u_n^2\right] \to 0$. Next we may assume

$$\int \left[|\nabla u_n|^2 + \eta(x) u_n^2\right] \to l, \qquad l \geqslant 0.$$

$$\int |u_n|^6 \to l, \qquad n \to \infty.$$

So $I(u_n) \to \frac{1}{3} l, n \to \infty$. In view of $c > 0$, then $l > 0$, by the definition of S,

$$S \leqslant \frac{\int \left[|\nabla u_n|^2 + \eta(x) u_n^2\right]}{\left(\int |u_n|^6\right)^{\frac{1}{3}}} \to l^{\frac{2}{3}},$$

so one has

$$c = \left(\frac{1}{2} - \frac{1}{6}\right) l \geqslant \frac{1}{3} S^{\frac{3}{2}}, \tag{3.24}$$

which makes a controdiction with the lemma. □

4. The Proof of Theorem 2.2

We can observe that as in [3], if $q = 4$, in the equation (3.23) one can stress the parameter choosing $\mu = \varepsilon^{-\delta}$, $\delta > 0$, then to get equation (3.19), the rest proof of Theorem 2.2 is similar to proof of Theorem 2.1.

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