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GROUND STATES SOLUTIONS FOR A MODIFIED FRACTIONAL SCHRÖDINGER EQUATION WITH A GENERALIZED CHOQUARD NONLINEARITY

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Abstract. In this paper, using variational methods, we study the existence of ground states solutions to the modified fractional Schrödinger equations with a generalized Choquard nonlinearity.

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1. Introduction

In the present paper, we investigate the existence of ground states solutions for a modified fractional Schrödinger equation with a generalized Choquard nonlinearity

(1.1)

$$(-\triangle)^s u + \mu V(x)u + 2\left[(-\triangle)^s u^2\right] u = (I_\lambda * F(u)) f(u) + \frac{|u|^{22^*_s(\beta)-2}u}{|x|^\beta}, \ x \in \mathbb{R}^{\mathbb{N}},$$

where $\mathbb{N} \geq 3$, $s \in (0,1)$, $0 \leq \beta < 2s < \mathbb{N}$, μ is positive constant, $2^*_s(\beta) = \dfrac{2(n-\beta)}{n-2s}$ is the critical β-fractional Sobolev exponent, V(x) is a given potential, $f \in C(\mathbb{R}, \mathbb{R})$ and $F \in C(\mathbb{R}, \mathbb{R})$ with $F(u) = \int_0^u f(t)dt$, $I_\lambda(x) = |x|^{-\lambda}$ is the Rieze potential of order $\lambda \in (0, N)$ and $(-\triangle)^s$ denotes the fractional Laplacian of order s is defined as

$$(-\triangle)^s \varphi(x) = 2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} dy, \qquad x \in \mathbb{R}^{\mathbb{N}},$$

with $\varphi \in C_0^\infty(\mathbb{R}^{\mathbb{N}})$ and $B_\epsilon(x)$ denotes the ball of $\mathbb{R}^{\mathbb{N}}$ centered at $x \in \mathbb{R}^N$ and radius $\epsilon > 0$.

The study of existence and uniqueness of positive solutions for Choquard type equations attracted a lot of attention of researchers due to its vast applications in physical models [1]. Fractional Choquard equations and their applications is very interesting, we refer the readers to [2] –[11] and the references therein. The authors in [9], by using the Mountain Pass Theorem and the Ekeland variational principle obtained the existence of nonnegative solutions a Schrödinger-Choquard-

Kirchhoff-type fractional p-equation. Ma and Zhang [8] studied the fractional order Choquard equation and proved the existence and multiplicity of weak solutions. In [3], the authors investigated a class of Brézis-Nirenberg type problems of nonlinear Choquard equation involving the fractional Laplacian in bounded domain Ω. Wang and Yang [12] by using an abstract critical point theorem based on a pseudo-index related to the cohomological index studied the bifurcation results for the critical Choquard problems involving fractional p-Laplacian operator:

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2}u + \left(\int_\Omega \frac{|u|^{p^*_{\mu,s}}}{|x-y|^\mu} dy \right) |u|^{p^*_{\mu,s}-2}u, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary and λ is a real parameter. Also, in [13] – [15], the authors have studied the existence of multiple solutions for problem (1.2), when $p = 2$. For more works on the Brezis-Nirenberg type results on semilinear elliptic equations with fractional Laplacian, we refer to [16] – [17] and references therein.

On the other hand, Shao and Wang in [18] established the following Kirchhoff equations with Hardy-Littlewood-Sobolev critical nonlinearity:

$$\begin{cases} -\triangle u + V(x)u - u\triangle u^2 + \lambda \left(I_\alpha * |u|^p \right) |u|^{p-2}u = K(x)u^{-\gamma}, & x \in \mathbb{R}^N, \\ u > 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $\alpha \in (0, N)$, $\lambda > 0$ and I_α is a Riesz potential. Under suitable assumption on K and V, the author obtained the existence of positive solutions for problem (1.3).

Zhang and Ji [19] studied the following problem

$$-\triangle u + V(x)u - u\triangle u^2 = \left(I_\alpha * G(u) \right) g(u), \qquad x \in \mathbb{R}^N, \tag{1.4}$$

where $\alpha \in (0, N)$, I_α is a Riesz potential and $V : \mathbb{R}^N \to \mathbb{R}$ is radial potential, and established the existence of ground state solutions for problem (1.4) by using the variational method. For more results on equations with Hardy-Littlewood-Sobolev critical nonlinearity and nonlocal fractional problems, we refer to [20] – [31] and references therein.

Recently, the authors in [32] studied the existence of ground state solutions for the following modified fractional Schrödinger equations

$$(-\triangle)^\alpha u + \mu u + \kappa \left[(-\triangle)^\alpha u^2 \right] u = \sigma |u|^{p-1}u + |u|^{q-1}u, \quad x \in \mathbb{R}^N,$$

where $0 < \alpha < 1$, $\mu > 0$, $N \geq 2$, $\kappa > 0$, $2 < p+1 < q < \frac{2N}{N-2\alpha}$.

Motivated by the above works, in this paper, we would like to study the existence of ground state solutions for problem (1.1).

Throughout the paper, we get the following conditions:
(V_1) $V(x) \geq 0$, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\Omega := int\left(V^{-1}(0)\right)$ is non-empty with smooth boundary;
(V_2) There exists $M > 0$ such that meas$(x \in \mathbb{R}^{\mathbb{N}} | V(x) \leq M) < \infty$, where meas (.) denotes the Lebesgue measure;
(f_1) $f \in C(\mathbb{R}, \mathbb{R})$, $\lim_{t\to 0} \frac{f(t)}{t} = 0$;
(f_2) $\lim_{t\to\infty} \frac{f(t)}{t^{q-1}} = 0$ for some $\frac{2N-\lambda}{N} \leq q \leq \frac{2N-\lambda}{N-2s}$;
(f_3) There exists $\alpha \in (4, 22_s^*(\beta))$ that $0 < \alpha F(t) < tf(t)$, for all $t \in \mathbb{R}$.
Also, we introduce the following fractional Choquard equation:

$$(1.5)\quad \begin{cases} (-\triangle)^s u + 2\left[(-\triangle)^s u^2\right] u = (I_\lambda * F(u))\, f(u) + \frac{|u|^{22_\alpha^*(\beta)-2}u}{|x|^\beta}, & x \in \Omega, \\ u = 0, \qquad x \in \mathbb{R}^{\mathbb{N}} \setminus \Omega, \end{cases}$$

where Ω is defined in (V_1). The main results are as follows:

Theorem 1.1. *Let $0 < \mu < \min\{N, 4s\}$. Assume that $(f_1) - (f_3)$ and $(V_1) - (V_2)$ hold. Then there exists $\mu^* > 0$ such that* (1.1) *has a least a ground state solution for any $\mu > \mu^*$.*

Theorem 1.2. *Under the assumptions of Theorem 1.1, assume that u_{μ_n} be a ground state of problem (1.1) with $\mu_n \to \infty$. Then, up to a subsequence, $u_{\mu_n} \to u$ in $H^s(R^N)$ as $n \to \infty$. Moreover, u is a ground state solution of problem* (1.5).

The paper is organized as follows. In Section 2, we recall some basic definitions of fractional Sobolev space and Hardy-Littlewood-Sobolev Inequality, and we give some useful auxiliary lemmas. In Section 3, we give the proof of the main results.

2. Preliminaries

In this section, we present some preliminaries and lemmas that are useful to the proof to the main results. The fractional Sobolev space $H^s(\mathbb{R}^N)$ $(0 < s < 1)$ is defined by

$$H^s(\mathbb{R}^N) = \left\{\psi \in L^2(\mathbb{R}^N) : \| (-\triangle)^{\frac{s}{2}} \psi\|^2 < \infty\right\},$$

with the norm

$$\|\psi\|_{H^s(\mathbb{R}^N)} = \left(\|\psi\|_2^2 + \| (-\triangle)^{\frac{s}{2}} \psi\|^2\right)^{\frac{1}{2}},$$

where

$$\| (-\triangle)^{\frac{s}{2}} \psi\| = \left(\iint_{\mathbb{R}^{2N}} \frac{|\psi(x) - \psi(y)|^2}{|x-y|^{N+2s}} dxdy\right)^{\frac{1}{2}}.$$

The space $D^{s,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$[\psi]_{s,2} = \| (-\triangle)^{\frac{s}{2}} \psi\|.$$

Let S be the best Sobolev constant

$$S := \inf_{\psi\in D^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|\psi\|^2}{\left(\int_{\mathbb{R}^N} |\psi|^{2_s^*(\alpha)}dx\right)^{\frac{2}{2_s^*(\alpha)}}}. \tag{2.1}$$

Also, define the space

$$E = \left\{\psi \in H^s(\mathbb{R}^{\mathbb{N}})| \int_{\mathbb{R}^{\mathbb{N}}} \mu V(x)\psi^2 dx < +\infty\right\},$$

with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} \mu V(x)u^2dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy.$$

Let us recall the following results.

Lemma 2.1. *(see* [33, Lemma 1]*)* $(E, ||.||)$ *is a uniformly convex Banach space.*

Lemma 2.2 ([34])**.** *Suppose that* V *satisfies* (V_2) *and* $\mu^* > 0$ *be a fixed constant. Then the embedding* $E \hookrightarrow L^\nu(\mathbb{R}^{\mathbb{N}})$ *is continuous for all* $\mu > \mu^*$ *and* $\nu \in [2, 2_s^*(\beta))$*. Moreover, for any* $R > 0$ *and* $\nu \in [1, 2_s^*(\beta)]$ *the embedding* $E \hookrightarrow L^\nu(B_R(0))$ *is compact.*

Proof. The proof is similar to that of Lemma 1 in [34], so we omit it here.

Now, we state the following fractional Hardy-Sobolev inequality

Lemma 2.3. *(*[35, Lemma 2]*) Assume that* $\alpha \in [0, 2s]$ *with* $2s < N$*. Then there exists a positive constant* C *such that*

$$\left(\int_{\mathbb{R}^{\mathbb{N}}} \frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha}dx\right)^{\frac{1}{2_s^*(\alpha)}} \leq C \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{1}{2}} \text{ for every } u \in H^s(\mathbb{R}^N).$$

Lemma 2.4. *(Hardy-Littlewood-Sobolev Inequality,* [36, Theorem 4.3]*) Suppose that* $r, t \in (1, \infty)$*,* $\lambda \in (0, N)$ *with*

$$\frac{1}{t} + \frac{1}{r} + \frac{\lambda}{N} = 2.$$

So there exists a sharp constant $C(N, \lambda, r, t) > 0$ *such that*

$$\iint_{\mathbb{R}^{2N}} \frac{|\zeta(x)|.|\eta(y)|}{|x-y|^\lambda}dxdy \leq C(N, \lambda, r, t)\|\zeta\|_r\|\eta\|_t,$$

for all $\zeta \in L^r(\mathbb{R}^{\mathbb{N}})$ *and* $\eta \in L^t(\mathbb{R}^{\mathbb{N}})$*.*

If $F \in L^t(\mathbb{R}^{\mathbb{N}})$ for some $t > 1$ with $\frac{2}{t} + \frac{\lambda}{N} = 2$, then by Lemma 2.4,

$$\iint_{\mathbb{R}^{2N}} \frac{|F(u(x))|.|F(u(y))|}{|x-y|^\lambda}dxdy$$

is well defined.

We mean by a weak solution of (1.1), any $u \in E$ such that

$$\int_{\mathbb{R}^{\mathbb{N}}} (-\triangle)^{\frac{s}{2}} u.(-\triangle)^{\frac{s}{2}} \varphi dx + \int_{\mathbb{R}^{\mathbb{N}}} \mu V(x) u\varphi dx + 2\int_{\mathbb{R}^{\mathbb{N}}} (-\triangle)^{\frac{s}{2}} u^2.(-\triangle)^{\frac{s}{2}} u\varphi dx$$
$$= \int_{\mathbb{R}^{\mathbb{N}}} (I_\lambda * F(u)) f(u)\varphi dx + \int_{\mathbb{R}^{\mathbb{N}}} \frac{|u|^{22_s^*(\beta)-2}u.\varphi}{|x|^\beta} dx,$$

for any $\varphi \in E$. The energy function corresponding to (1.1) is

$$I_\mu(u) = \frac{1}{2}[u]_{s,2}^2 + \frac{\mu}{2}\int_{\mathbb{R}^{\mathbb{N}}} V(x)|u|^2 dx + \frac{1}{2}[u^2]_{s,2}^2 -$$
$$\frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x-y|^\lambda} dxdy - \frac{1}{22_s^*(\beta)}\int_{\mathbb{R}^{\mathbb{N}}} \frac{|u|^{22_s^*(\beta)}}{|x|^\beta},$$

and energy function corresponding to (1.5) is

$$I_0(u) = \frac{1}{2}[u]_{s,2}^2 + \frac{1}{2}[u^2]_{s,2}^2$$
$$-\frac{1}{2}\iint_{\Omega\times\Omega} \frac{F(u(x))F(u(y))}{|x-y|^\lambda} dxdy - \frac{1}{22_s^*(\beta)}\int_{\Omega} \frac{|u|^{22_s^*(\beta)}}{|x|^\beta}.$$

Set $X := \left\{\zeta \in E : \zeta^2 \in E\right\}$ with $\|\zeta\|_X = \|\zeta\|_E$ and

$$X_0 := \left\{\zeta \in H^s(\mathbb{R}^{\mathbb{N}}) : \zeta^2 \in H^s(\mathbb{R}^{\mathbb{N}}), \quad u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\}.$$

Now, we show that $X \neq \emptyset$. For simplicity, we assume N= 1. Let

$$u(x) := \begin{cases} \sqrt{|\sin(x)|} & x \in [1, 2\pi], \\ 0 & x \in \mathbb{R} \setminus [1, 2\pi]. \end{cases}$$

and

$$V(x) := \begin{cases} \frac{|x|-1}{|x|^3} & x \in \mathbb{R} \setminus (-1, 1), \\ 0 & x \in (-1, 1). \end{cases}$$

$$\begin{aligned} \iint_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}} dxdy &= \iint_{[1,2\pi]\times[1,2\pi]} \frac{|\sqrt{|\sin(x)|} - \sqrt{|\sin(y)|}|^2}{|x-y|^{1+2s}} dxdy \\ &\leq \iint_{[1,2\pi]\times[1,2\pi]} \frac{|\sqrt{|\sin(x)-\sin(y)|}|^2}{|x-y|^{1+2s}} dxdy \\ &\leq C_1 \iint_{[1,2\pi]\times[1,2\pi]} \frac{1}{|x-y|^{1+2s}} dxdy < \infty, \end{aligned}$$

where $C_1 \geq 0$ and

$$\int_{\mathbb{R}} \mu V(x)|u(x)|^2 dx \leq \int_{\mathbb{R}} \mu V(x) dx < \infty,$$

then $u(x) \in E$. In addition, we have

$$\begin{aligned} \iint_{\mathbb{R}^2} \frac{|u^2(x)-u^2(y)|^2}{|x-y|^{1+2s}} dxdy &= \iint_{[1,2\pi]\times[1,2\pi]} \frac{||\sin(x)| - |\sin(y)||^2}{|x-y|^{1+2s}} dxdy \\ &\leq C_2 \iint_{[1,2\pi]\times[1,2\pi]} \frac{1}{|x-y|^{1+2s}} dxdy < \infty, \end{aligned}$$

where $C_2 \geq 0$ and

$$\int_{\mathbb{R}} \mu V(x)|u^2(x)|^2 dx \leq \int_{\mathbb{R}} \mu V(x) dx < \infty,$$

then $u^2(x) \in E$ and $u(x) \in X$. Then $X \neq \emptyset$.

Also, $I_\mu(u)$ is well defined on X and $I_0(u)$ is well defined on X_0. Under the assumation (V_1) asnd (V_2), I_μ, I_0 are well defined and $I_\mu, I_0 \in C^1(X, \mathbb{R}^N)$.

Let

$$\mathbb{J}(u) = \iint_{\mathbb{R}^{2N}} \frac{|u^2(x) - u^2(y)|^2}{|x-y|^{N+2s}} dxdy.$$

We have

(2.2)

$$\prec \mathbb{J}'(u), v \succ = \frac{d}{dt}\mathbb{J}(u+tv) \mid_{t=0} = \frac{d}{dt} \iint_{\mathbb{R}^{2N}} \frac{|(u(x)+tv(x))^2 - (u(y)+tv(y))^2|^2}{|x-y|^{N+2s}} dxdy$$

$$= 2 \iint_{\mathbb{R}^{2N}} \frac{\Big((u(x)+tv(x))^2 - (u(y)+tv(y))^2\Big)}{|x-y|^{N+2s}} \times \Big(2(u(x)+tv(x))v(x) - 2(u(y)+tv(y))v(y)\Big) dxdy \mid_{t=0} \tag{2.3}$$

$$= 4 \iint_{\mathbb{R}^{2N}} \frac{\left(u^2(x) - u^2(y)\right)\left(u(x)v(x) - u(y)v(y)\right)}{|x-y|^{N+2s}} dxdy.$$

So by (2.2), we can easily check that

$$\begin{aligned}
\left\langle I_\mu'(u), \varrho \right\rangle &= \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\varrho(x)-\varrho(y))}{|x-y|^{N+2s}} dxdy + \int_{\mathbb{R}^{\mathbb{N}}} \mu V(x) u(x) \varrho(x) dx \\
&\quad + 2 \iint_{\mathbb{R}^{2N}} \frac{(u^2(x)-u^2(y))(u(x)\varrho(x) - u(y)\varrho(y))}{|x-y|^{N+2s}} dxdy \\
&\quad - \iint_{\mathbb{R}^{2N}} \frac{F(u(y)) f(u(x)) \varrho(x)}{|x-y|^\lambda} dxdy - \int_{\mathbb{R}^{\mathbb{N}}} \frac{|u|^{22_s^*(\beta)-2} u(x) \varrho(x)}{|x|^\beta} dx,
\end{aligned}$$

for all $u, \varrho \in X$ and

$$\begin{aligned}
\left\langle I_0'(u), \varrho \right\rangle &= \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\varrho(x)-\varrho(y))}{|x-y|^{N+2s}} dxdy \\
&\quad + 2 \iint_{\mathbb{R}^{2N}} \frac{(u^2(x)-u^2(y))(u(x)\varrho(x) - u(y)\varrho(y))}{|x-y|^{N+2s}} dxdy \\
&\quad - \iint_{\Omega\times\Omega} \frac{F(u(y)) f(u(x)) \varrho(x)}{|x-y|^\lambda} dxdy - \int_{\Omega} \frac{|u|^{22_s^*(\beta)-2} u(x) \varrho(x)}{|x|^\beta} dx,
\end{aligned}$$

for all $u, \varrho \in X_0$.

Lemma 2.5. *Assume that (f_1) and (f_2), we have*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(u(y))}{|x-y|^\lambda} f(u(x)) u(x) dxdy \right| \leq C([u]_{s,2}^4 + [u]_{s,2}^{2q}), \tag{2.4}$$

and

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(y))F(u(x))}{|x-y|^\lambda}dxdy\right|\le C([u]_{s,2}^4+[u]_{s,2}^{2q}). \tag{2.5}$$

Proof. The proof is similar to that of Lemma 2.5 in [37], so we omit it here.

Lemma 2.6. *Assume that* $\{u_n\}\subset E$ *such that* $u_n\rightharpoonup u$ *in* E. *From* $(f_1),(f_2)$ *and* $0<\lambda<\min\{N,4S\}$, *we have*

$$\int_{\mathbb{R}^{\mathbb{N}}}(I_\lambda * F(u_n))F(u_n)dx\to\int_{\mathbb{R}^{\mathbb{N}}}(I_\lambda * F(u))F(u)dx,$$
$$\int_{\mathbb{R}^{\mathbb{N}}}(I_\lambda * F(u_n))f(u_n)\varphi dx\to\int_{\mathbb{R}^{\mathbb{N}}}(I_\lambda * F(u))f(u)\varphi dx.$$

as $n\to\infty$.

Proof. The proof is similar to that of the proof of Lemma 2.6 in [37], so we omit it here. Set

$$m_\mu:=\inf_{u\in\Sigma}I_\mu(u),\qquad m_0:=\inf_{u\in\Sigma_0}I_0(u),$$

where

$$\Sigma:=\left\{u\in X\setminus\{0\}\mid <I_\mu^{'}(u),u>=0\right\},\qquad \Sigma_0:=\left\{u\in X_0\setminus\{0\}\mid <I_0^{'}(u),u>=0\right\}.$$

We know that to prove our main results, we should check that m_μ is achieved by a critical point of I_μ for $\mu>\mu^*$.

Lemma 2.7. $\Sigma_0\neq\varnothing$.

Proof. Let $u_0\in X\setminus\{0\}$ with $u_0\ge 0$ and $\kappa(t)=\zeta\left(\frac{tu_0}{[u_0]_{s,2}}\right)$, where

$$\zeta(u)=\iint_{\Omega\times\Omega}\frac{F(u(y))F(u(x))}{|x-y|^\lambda}dxdy.$$

From (f_3), we have

$$\frac{\alpha}{t}\le\frac{\kappa^{'}(t)}{\kappa(t)},\quad\forall\, t>0.$$

Consequently, by integrating from the above inequality over $[1,t[u_0]_{s,2}]$ with $t>\frac{1}{[u_0]_{s,2}}$, one can get

$$\zeta(tu_0)\ge\zeta\left(\frac{u_0}{[u_0]_{s,2}}\right)t^\alpha[u_0]_{s,2}^\alpha.$$

So, we get

$$I_0(t_0u_0)\le\frac{t_0^2}{2}[u_0]_{s,2}^2+\frac{t_0^4}{2}[u_0^2]_{s,2}^2-\frac{\lambda}{2}\zeta(\frac{u_0}{[u_0]_{s,2}})t^\alpha[u_0]_{s,2}^\alpha,$$

since $\alpha>4$, if $t_0\to+\infty$, we have $I_0(t_0u_0)\to-\infty$.

On the other hand,

$$
\begin{aligned}
I_0(t_0u_0) &= \frac{t_0^2}{2}[u_0]_{s,2}^2 + \frac{t_0^4}{2}[u_0^2]_{s,2}^2 - \frac{1}{2}\iint_{\Omega\times\Omega}\frac{F(t_0u_0(x))F(t_0u_0(y))}{|x-y|^\lambda}dxdy \\
&\quad - \frac{t_0^{22_s^*(\beta)}}{22_s^*(\beta)}\int_\Omega \frac{|u|^{22_s^*(\beta)}}{|x|^\beta} \geq \frac{t_0^2}{2}[u_0]_{s,2}^2 + \frac{t_0^4}{2}[u_0^2]_{s,2}^2 \\
&\quad - C_1\left(t_0^4[u_0]_{s,2}^4 + t_0^{2q}[u_0]_{s,2}^{2q}\right) - C_2 t_0^{22_s^*(\beta)}[u_0^2]_{s,2}^{2_s^*(\beta)},
\end{aligned}
$$

which implies that for small $t_0 > 0$, $I_0(t_0u_0) > 0$. Then, there exists $t > 0$ such that $\frac{d}{dt}|_{t_0=t} I_0(tu_0) = 0$, means, $tu_0 \in \Sigma_0$, then we have the conclusion. □

Lemma 2.8. *There exists $K > 0$ such that $m_\mu \geq K$.*

Proof. We divide the proof into the following three steps.

Step 1: $\Sigma_0 \subset \Sigma$ and $m_0 \geq m_\mu$.

For any $u \in \Sigma_0$, by the definition of Ω, one has

$$\int_{\mathbb{R}^N} \mu V(x)|u|^2 dx = 0.$$

Consequently,

$$< I_\mu^{'}(u), u > = < I_0^{'}(u), u > + \int_{\mathbb{R}^N} \mu V(x)|u|^2 dx,$$

hence, $u \in \Sigma$ and $\Sigma_0 \subset \Sigma$, $\Sigma \neq \varnothing$. Similarly, we can prove that $I_\mu(u) = I_0(u)$, and then we get

$$m_\mu = \inf_{u\in\Sigma} I_\mu(u) \leq \inf_{u\in\Sigma_0} I_\mu(u) = \inf_{u\in\Sigma_0} I_0(u) = m_0.$$

Step 2: m_μ is bounded from below.

From (f_3), for any $u \in \Sigma$, we get

$$
\begin{aligned}
I_\mu(u) &= I_\mu(u) - \frac{1}{\alpha} < I_\mu^{'}(u), u > \\
&= \left(\frac{1}{2} - \frac{1}{\alpha}\right)[u]_{s,2}^2 + \left(\frac{1}{2} - \frac{1}{\alpha}\right)\int_{\mathbb{R}^N} \mu V(x)|u|^2 dx \\
&+ \left(\frac{1}{2} - \frac{2}{\alpha}\right)[u^2]_{s,2}^2 - \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{F(u(y))F(u(x))}{|x-y|^\lambda}dxdy \\
&+ \frac{1}{\alpha}\iint_{\mathbb{R}^{2N}} \frac{F(u(y))f(u(x))u(x)}{|x-y|^\lambda}dxdy - \left(\frac{1}{22_s^*(\beta)} - \frac{1}{\alpha}\right)\int_{\mathbb{R}^N} \frac{|u|^{22_s^*(\beta)}}{|x|^\beta}dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\alpha}\right)[u]_{s,2}^2 + \left(\frac{1}{2} - \frac{1}{\alpha}\right)\int_{\mathbb{R}^N} \mu V(x)|u|^2 dx \\
&+ \left(\frac{1}{2} - \frac{2}{\alpha}\right)[u^2]_{s,2}^2 - \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{F(u(y))F(u(x))}{|x-y|^\lambda}dxdy
\end{aligned}
$$

$$+\frac{1}{2\alpha}\iint_{\mathbb{R}^{2N}}\frac{F(u(y))f(u(x))u(x)}{|x-y|^{\lambda}}dxdy-\left(\frac{1}{22_s^*(\beta)}-\frac{1}{\alpha}\right)\int_{\mathbb{R}^{\mathbb{N}}}\frac{|u|^{22_s^*(\beta)}}{|x|^{\beta}}dx \tag{2.6}$$
$$=\left(\frac{1}{2}-\frac{1}{\alpha}\right)[u]_{s,2}^2+\left(\frac{1}{2}-\frac{1}{\alpha}\right)\int_{\mathbb{R}^{\mathbb{N}}}\mu V(x)|u|^2dx+\left(\frac{1}{2}-\frac{2}{\alpha}\right)[u^2]_{s,2}^2$$
$$-\left(\frac{1}{22_s^*(\beta)}-\frac{1}{\alpha}\right)\int_{\mathbb{R}^{\mathbb{N}}}\frac{|u|^{22_s^*(\beta)}}{|x|^{\beta}}dx,$$

since $\alpha\in(4,22_s^*(\beta))$, then $(\frac{1}{2}-\frac{1}{\alpha})>0,(\frac{1}{\alpha}-\frac{1}{22_s^*(\beta)})>0$, consequently, $I_\mu(u)\geq 0$. This result implies that $m_\mu\geq 0$.

Step 3: m_μ have positive uniform bounded from below.

Let $\{u_n\}$ be a minimizing sequence of m, then $I_\mu(u_n)\to m$ and $I_\mu^{'}(u_n)\to 0$. According to the proof of the (2.6), we have

$$m_0+o_n(1)\geq m_\mu+o_n(1)$$
$$\geq\left(\frac{1}{2}-\frac{1}{\alpha}\right)[u_n]_{s,2}^2+\left(\frac{1}{2}-\frac{1}{\alpha}\right)\int_{\mathbb{R}^{\mathbb{N}}}\mu V(x)|u_n|^2dx+\left(\frac{1}{2}-\frac{2}{\alpha}\right)[u_n^2]_{s,2}^2$$
$$-\left(\frac{1}{22_s^*(\beta)}-\frac{1}{\alpha}\right)\int_{\mathbb{R}^{\mathbb{N}}}\frac{|u_n|^{22_s^*(\beta)}}{|x|^{\beta}}dx\geq\left(\frac{1}{2}-\frac{1}{\alpha}\right)[u_n]_{s,2}^2 \tag{2.7}$$
$$+\left(\frac{1}{2}-\frac{1}{\alpha}\right)\int_{\mathbb{R}^{\mathbb{N}}}\mu V(x)|u_n|^2dx.$$

Thus

$$m_0+o_n(1)\geq m_\mu+o_n(1)\geq C_1||u_n||^2, \tag{2.8}$$

where $C_1=(\frac{1}{2}-\frac{1}{\alpha})$. From fractional Hardy-Sobolev inequality and lemma 2.5, there exist two constants $C_2,C_3>0$ such that

$$\|u_n\|^2\leq\|u_n\|^2+[u_n^2]_{s,2}^2$$
$$=\iint_{\mathbb{R}^{2N}}\frac{F(u_n(y))f(u_n(x))\varphi}{|x-y|^{\lambda}}dxdy+\int_{\mathbb{R}^{\mathbb{N}}}\frac{|u_n|^{22_s^*(\beta)-2}u_n\varphi}{|x|^{\beta}}dx$$
$$\leq C_2([u_n]_{s,2}^4+[u_n]_{s,2}^{2q})+C_3[u_n]_{s,2}^{22_s^*(\beta)}$$
$$\leq C_2(\|u_n\|^4+\|u_n\|^{2q})+C_3\|u_n\|^{22_s^*(\beta)}.$$

So, we may choose a constant $C_4>0$ such that

$$\|u_n\|^2\geq C_4. \tag{2.9}$$

From (2.8) and (2.9), there exist $K:=C_1\times C_4>0$, such that

$$m_\mu\geq\|u_n\|^2\geq K.$$

Therefore, we have the conclusion. □

3. Proof of the main theorems

In this section, we prove our main results.

Proof of Theorem 1.1. Fix $\mu > \mu^*$ and take a sequence $\{u_n\} \subset \Sigma$, that is $I_\mu(u_n) \to m_\mu$. Then, by (2.8), $\{u_n\}$ is bounded in X. Hence, $u_n \rightharpoonup u$, $u_n^2 \rightharpoonup u^2$ in E up to subsequence, and thus by Lemma 2.2,

$$\begin{cases} u_n \to u, \;\; u_n^2 \to u^2 \;\text{ in } L^s_{loc}(\mathbb{R}^{\mathbb{N}}) \;\; (1 \le s < 2_s^*(\beta)), \\ u_n \to u, \;\; \text{a.e. in } \mathbb{R}^N, \\ \frac{|u_n|}{|x|^\beta} \to \frac{|u|}{|x|^\beta} \quad in \quad L^r(\mathbb{R}^{\mathbb{N}}, \frac{dx}{|x|^\beta}) \quad for\ 2 \le r < 2_s^*(\beta)\ and\ 0 \le \beta < 2s. \end{cases} \tag{3.1}$$

Let $\psi \in H^s(\mathbb{R}^{\mathbb{N}})$ and we define a linear functional on X as follows

$$B_\psi(\varphi) = \iint_{\mathbb{R}^{2N}} \frac{(\psi^2(x) - \psi^2(y))(\psi(x)\varphi(x) - \psi(y)\varphi(y))}{|x-y|^{N+2s}} dxdy, \;\; \forall\, \varphi \in X.$$

Hence, one has

$$\lim_{n\to\infty} B_u(u_n - u) = 0. \tag{3.2}$$

Let $\xi \in X$ be fixed and Φ_υ be the linear functional on X defined by

$$\Phi_\xi(\upsilon) = \iint_{\mathbb{R}^{\mathbb{N}}} \frac{(\xi(x) - \xi(y))(\upsilon(x) - \upsilon(y))}{|x-y|^{N+2s}} dxdy, \;\; \forall\, \upsilon \in X.$$

Since $I_\mu^{'}(u_n) \to 0$, one can get

$$\lim_{n\to\infty} < I_\mu^{'}(u_n) - I_\mu^{'}(u), u_n - u >= 0.$$

Consequently,

$$\begin{aligned} o(1) =< I_\mu^{'}(u_n) - I_\mu^{'}(u), u_n - u > &= \Phi_{u_n}(u_n - u) - \Phi_u(u_n - u) + 2B_{u_n}(u_n - u) \\ + \int_{\mathbb{R}^{\mathbb{N}}} \mu V(x)|u_n(x) - u(x)|^2 dx &- \iint_{\mathbb{R}^{2N}} \frac{F(u_n(y))f(u_n(x))(u_n(x) - u(x))}{|x-y|^\lambda} dxdy \\ &+ \iint_{\mathbb{R}^{2N}} \frac{F(u(y))f(u(x))(u_n(x) - u(x))}{|x-y|^\lambda} dxdy \\ &- \int_{\mathbb{R}^{\mathbb{N}}} [\frac{|u_n|^{22_s^*(\beta)-2}u_n - |u|^{22_s^*(\beta)-2}u}{|x|^\beta}](u_n - u)dx. \end{aligned}$$

From Lemma 2.6, we have

$$\iint_{\mathbb{R}^{2N}} \frac{(F(u_n(y))f(u_n(x)) - F(u(y))f(u(x)))(u_n(x) - u(x))}{|x-y|^\lambda} dxdy \to 0, \;\text{as}\, n \to \infty. \tag{3.3}$$

Also, in view of (3.1), we get

$$\int_{\mathbb{R}^{\mathbb{N}}} \mu V(x)|u_n(x) - u(x)|^2 dx \to 0, \quad \text{as} \quad n \to \infty. \tag{3.4}$$

By similare method of proof Lemma 3.4. in [37], we have

$$\frac{|u_n|^{22_s^*(\beta)}}{|x|^\beta} \to \frac{|u|^{22_s^*(\beta)}}{|x|^\beta}. \tag{3.5}$$

Moreover, from (3.5) and Brezis-Lieb Lemma [38], we get

(3.6)
$$\int_{\mathbb{R}^{\mathbb{N}}} \frac{|u_n-u|^{22_s^*(\alpha)}}{|x|^\beta}dx = \int_{\mathbb{R}^{\mathbb{N}}} \frac{|u_n|^{22_s^*(\alpha)}}{|x|^\beta}dx - \int_{\mathbb{R}^{\mathbb{N}}} \frac{|u|^{22_s^*(\alpha)}}{|x|^\beta}dx + o(1) \to 0, \text{ as } n \to \infty.$$

So, by (3.6) and the Hölder inequality, we have

$$\int_{\mathbb{R}^{\mathbb{N}}} \left[\frac{|u_n|^{22_s^*(\beta)-2}u_n}{|x|^\beta} - \frac{|u|^{22_s^*(\beta)-2}u}{|x|^\beta}\right](u_n-u)dx \to 0 \quad as\ n \to \infty. \tag{3.7}$$

Hence, in view of the Hölder inequality, one can get

$$\Phi_{u_n}(u_n-u) - \Phi_u(u_n-u) \geq ([u_n]_{s,2} - [u]_{s,2})^2 \geq 0 \tag{3.8}$$

From (3.3) – (3.8) and $B_{u_n}(u_n - u) \geq 0$, we have $\|u_n\| \to \|u\|$. Since X uniformly convex Banach space, then the weak convergence and norm convergence imply strong convergence. In view of $I_\mu \in C(X, R)$, $I_\mu(u) = m_\mu$ and $I^{'}(u) = 0$. Hence, we have the conclusion. □

Proof of Theorem 1.2. Take u_{μ_n} be a ground state of I_{μ_n} as $\mu_n \to \infty$, that is, $I_{\mu_n}(u_{\mu_n}) = m_{\mu_n}$ and $I^{'}_{\mu_n}(u_{\mu_n}) = 0$. For notion simplicity, we denote u_{μ_n} by u_n. We may suppose that $\mu_n > \mu^*$ for all n without loss of generality. In view of (2.7), we get

$$m_0 \geq m_{\mu_n} \geq (\frac{1}{2} - \frac{1}{\alpha})[u]_{s,2}^2 + (\frac{1}{2} - \frac{1}{\alpha})\int_{\mathbb{R}^{\mathbb{N}}} \mu V(x)|u|^2 dx.$$

In view of Lemma 2.2, we can get

$$\begin{cases} u_n \rightharpoonup u, u_n^2 \rightharpoonup u^2, \text{ in } H^s(\mathbb{R}^{\mathbb{N}}), \\ u_n \to u, \ u_n^2 \to u^2 \text{ in } L^s_{loc}(\mathbb{R}^{\mathbb{N}}) \ (1 \leq s < 2_s^*(\beta)), \\ u_n \to u, \text{ a.e. in } \mathbb{R}^N, \\ \frac{|u_n|}{|x|^\beta} \to \frac{|u|}{|x|^\beta} \quad in \quad L^r(\mathbb{R}^{\mathbb{N}}, \frac{dx}{|x|^\beta}) \quad for\ 2 \leq r < 2_s^*(\beta)\ and\ 0 \leq \beta < 2s. \end{cases} \tag{3.9}$$

We divide the proof into the following three steps:

Step 1: $u(x) = 0$ a.e in $\mathbb{R}^{\mathbb{N}} \setminus \Omega$.

By (2.7), we get

$$\int_{\mathbb{R}^{\mathbb{N}}} V(x)|u_n|^2 dx \leq \frac{Cm_0}{\mu_n} \to 0, \quad \text{as } n \to \infty.$$

Also, the Fatou's Lemma implies that

$$\int_{\mathbb{R}^{\mathbb{N}} \setminus \Omega} V(x)|u|^2 dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^{\mathbb{N}}} V(x)|u_n|^2 dx = 0.$$

Hence, we have $u(x) = 0$ a.e in $\mathbb{R}^{\mathbb{N}} \setminus \Omega$.

Step 2: u is a critical point of I_0. Since $I^{'}_{\mu_n}(u_n) = 0$, we have

$$\iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\zeta(x)-\zeta(y))}{|x-y|^{N+2s}}dxdy + \int_{\mathbb{R}^{\mathbb{N}}} \mu_n V(x) u_n \zeta(x)dx$$
$$+2\iint_{\mathbb{R}^{2N}} \frac{(u_n^2(x)-u_n^2(y))(u_n(x)\zeta(x)-u_n(y)\zeta(y))}{|x-y|^{N+2s}}dxdy$$
$$-\iint_{\mathbb{R}^{2N}} \frac{F(u_n(y))f(u_n(x))\zeta(x)}{|x-y|^{\lambda}}dxdy - \int_{\mathbb{R}^{\mathbb{N}}} \frac{|u_n|^{22_s^*(\beta)-2}u_n\zeta(x)}{|x|^{\beta}}dx = 0,$$

for all $\zeta \in H^s(\mathbb{R}^{\mathbb{N}})$. Now, in view of (3.9) and $V(x)=0$ in Ω,

$$\iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\zeta(x)-\zeta(y))}{|x-y|^{N+2s}}dxdy \to \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\zeta(x)-\zeta(y))}{|x-y|^{N+2s}}dxdy, \tag{3.10}$$

$$\iint_{\mathbb{R}^{2N}} \frac{(u_n^2(x)-u_n^2(y))(u_n(x)\zeta(x)-u_n(y)\zeta(y))}{|x-y|^{N+2s}}dxdy \to$$
$$\iint_{\mathbb{R}^{2N}} \frac{(u^2(x)-u^2(y))(u(x)\zeta(x)-u(y)\zeta(y))}{|x-y|^{N+2s}}dxdy, \tag{3.11}$$

as $n \to \infty$, and

$$\lim_{n\to\infty} \int_{\mathbb{R}^{\mathbb{N}}} \mu_n V(x) u_n \zeta(x)dx = 0, \tag{3.12}$$

for all $\varphi \in H^s(\mathbb{R}^{\mathbb{N}})$. From Lemma 2.6, we have

$$\iint_{\mathbb{R}^{2N}} \frac{F(u_n(y))f(u_n(x))\zeta(x)}{|x-y|^{\lambda}}dxdy \to \iint_{\mathbb{R}^{2N}} \frac{F(u(y))f(u(x))\zeta(x)}{|x-y|^{\lambda}}dxdy,\ \forall \zeta \in H^s(\mathbb{R}^{\mathbb{N}}), \tag{3.13}$$

similarly to (3.7), we get

$$\int_{\mathbb{R}^{\mathbb{N}}} \frac{|u_n|^{22_s^*(\beta)-2}u_n\zeta(x)}{|x|^{\beta}}dx \to \int_{\mathbb{R}^{\mathbb{N}}} \frac{|u|^{22_s^*(\beta)-2}u\zeta(x)}{|x|^{\beta}}dx,\ \forall \zeta \in H^s(\mathbb{R}^{\mathbb{N}}). \tag{3.14}$$

Then, $(3.10)-(3.14)$ and step 1 imply that

$$\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\zeta(x)-\zeta(y))}{|x-y|^{N+2s}}dxdy$$
$$+2\iint_{\mathbb{R}^{2N}} \frac{(u^2(x)-u^2(y))(u(x)\zeta(x)-u(y)\zeta(y))}{|x-y|^{N+2s}}dxdy$$
$$-\iint_{\Omega\times\Omega} \frac{F(u(y))f(u(x))\zeta(x)}{|x-y|^{\lambda}}dxdy - \int_{\Omega} \frac{|u|^{22_s^*(\beta)-2}u\zeta}{|x|^{\beta}}dx = 0,\ \forall \zeta \in H^s(\mathbb{R}^{\mathbb{N}}),$$

which implies that u is a critical point of I_0.

Step 3: $u_n \to u$ in $L^s(\mathbb{R}^{\mathbb{N}})$ for $2 \le s < 2_s^*(\beta)$.

From (3.9), by decay of the lebesgue integral, there exist $R>0$, such that

$$\int_{\mathbb{R}^{\mathbb{N}}\setminus B_R(0)} |u(x)|^2 dx < \epsilon. \tag{3.15}$$

Let $\omega_1 := \left\{x \in \mathbb{R}^{\mathbb{N}} : |x| > R' \quad \text{and} \quad V(x) \le M\right\}$,

$$\omega_2 := \left\{x \in \mathbb{R}^{\mathbb{N}} : |x| > R' \quad \text{and} \quad V(x) > M\right\}.$$

From (V_2), we have

$$\lim_{R'\to\infty} \text{meas}(\omega_1(R')) = 0. \tag{3.16}$$

By the Hölder inequality and the Sobolev embedding theoream, we can get

$$\begin{aligned}\int_{\omega_1(R')} |u_n(x)|^2 dx &\le \left(\text{meas}(\omega_1(R'))\right)^{\frac{2s-\beta}{N-\beta}} \left(\int_{\omega_1(R')} |u_n(x)|^{2_s^*(\beta)} dx\right)^{\frac{2}{2_s^*(\beta)}} \\ &\le C\left(\text{meas}(\omega_1(R'))\right)^{\frac{2s-\beta}{N-\beta}}. \end{aligned} \tag{3.17}$$

On the other hand

$$\int_{\omega_2(R')} |u_n(x)|^2 dx \le \frac{1}{\mu M}\int_{\omega_2(R')} \mu M |u_n(x)|^2 dx \le \frac{C}{\mu M}. \tag{3.18}$$

From (3.15) – (3.18), for any $\varepsilon > 0$, we may choose $\mu_0 > 0$ and $R' > 0$ such that

$$\int_{\mathbb{R}^{\mathbb{N}}\setminus B_{R'}(0)} |u_n(x)|^2 dx < \epsilon \quad \text{for} \quad \mu \ge \mu_0. \tag{3.19}$$

Take $R_0 = \max\{R, R'\}$,

$$\begin{aligned}\int_{\mathbb{R}^{\mathbb{N}}} |u_n - u|^2 dx &= \int_{B^c_{R_0}(0)} |u_n - u|^2 dx + \int_{B_{R_0}(0)} |u_n - u|^2 dx \\ &\le 2\int_{B^c_{R_0}(0)} |u_n|^2 dx + 2\int_{B^c_{R_0}(0)} |u|^2 dx + \int_{B_{R_0}(0)} |u_n - u|^2 dx \\ &\le 4\varepsilon + \int_{B_{R_0}(0)} |u_n - u|^2 dx.\end{aligned}$$

Also, by Lemma 2.2, we get $u_n \to u$ in $L^2(\mathbb{R}^{\mathbb{N}})$ as $n \to \infty$. Since $u_n \rightharpoonup u$ in E and $u_n \to u$ in $L^2(\mathbb{R}^{\mathbb{N}})$, one can get $u_n \to u$ in $L^s(\mathbb{R}^{\mathbb{N}})$ for $2 \le s < 2_s^*(\beta)$.

Step 4: m_0 is achieved by u. Moreover, $u_n \to u$ in $H^s(\mathbb{R}^{\mathbb{N}})$.

By the lower semi-continuity, we have

$$\liminf_{n\to\infty} [u_n]^2_{s,2} \ge [u]^2_{s,2}, \qquad \liminf_{n\to\infty} [u_n^2]^2_{s,2} \ge [u^2]^2_{s,2}. \tag{3.20}$$

In the other hand, by similar method in (2.6), we can obtain

$$\begin{aligned} m_0 &\ge \lim_{n\to\infty} m_{\mu_n} = \lim_{n\to\infty}\left(I_{\mu_n}(u_n) - \frac{1}{\alpha} < I'_{\mu_n}(u_n), u_n >\right) \\ &= \lim_{n\to\infty}\Bigg\{\left(\frac{1}{2} - \frac{1}{\alpha}\right)[u_n]^2_{s,2} + \left(\frac{1}{2} - \frac{1}{\alpha}\right)\int_{\mathbb{R}^{\mathbb{N}}} \mu_n V(x)|u_n|^2 dx \\ &+ \left(\frac{1}{2} - \frac{2}{\alpha}\right)[u_n^2]^2_{s,2} - \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{F(u_n(x))F(u_n(y))}{|x-y|^{\lambda}} dxdy \\ &+ \frac{1}{\alpha}\iint_{\mathbb{R}^{2N}} \frac{F(u_n(y))f(u_n(x))u_n(x)}{|x-y|^{\lambda}} dxdy - \left(\frac{1}{22_s^*(\beta)} - \frac{1}{\alpha}\right)\int_{\mathbb{R}^{\mathbb{N}}} \frac{|u_n|^{22_s^*(\beta)}}{|x|^{\beta}} dx\Bigg\} \\ &\ge \left(\frac{1}{2} - \frac{1}{\alpha}\right)[u]^2_{s,2} + \left(\frac{1}{2} - \frac{2}{\alpha}\right)[u^2]^2_{s,2} - \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x-y|^{\lambda}} dxdy \end{aligned}$$

$$+\frac{1}{\alpha}\iint_{\mathbb{R}^{2N}}\frac{F(u(y))f(u(x))u(x)}{|x-y|^{\lambda}}dxdy-\left(\frac{1}{22_s^*(\beta)}-\frac{1}{\alpha}\right)\int_{\mathbb{R}^{\mathbb{N}}}\frac{|u|^{22_s^*(\beta)}}{|x|^{\beta}}dx$$
$$=\left(\frac{1}{2}-\frac{1}{\alpha}\right)[u]_{s,2}^2+\left(\frac{1}{2}-\frac{2}{\alpha}\right)[u^2]_{s,2}^2-\frac{1}{2}\iint_{\Omega\times\Omega}\frac{F(u(x))F(u(y))}{|x-y|^{\lambda}}dxdy$$
$$+\frac{1}{\alpha}\iint_{\Omega\times\Omega}\frac{F(u(y))f(u(x))u(x)}{|x-y|^{\lambda}}dxdy-\left(\frac{1}{22_s^*(\beta)}-\frac{1}{\alpha}\right)\int_{\Omega}\frac{|u|^{22_s^*(\beta)}}{|x|^{\beta}}dx=I_0(u)\geq m_0.$$

which implies that $I_0(u)=m_0$, $\lim_{n\to\infty}m_{\mu_n}=m_0$, and

$$\liminf_{n\to\infty}[u_n]_{s,2}^2=[u]_{s,2}^2,\qquad \liminf_{n\to\infty}[u_n^2]_{s,2}^2=[u^2]_{s,2}^2. \tag{3.21}$$

By step 3 and (3.21), we have $\|u_n\|_{H^s(\mathbb{R}^N)}\to\|u\|_{H^s(\mathbb{R}^N)}$. This together with the fact that $u_n\rightharpoonup u$ in $H^s(\mathbb{R}^N)$, we get $u_n\to u$ in $H^s(\mathbb{R}^N)$. □

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