ADHESIVE INTERACTION OF A PIECEWISE-HOMOGENEOUS ORTHOTROPIC PLATE WITH AN ELASTIC BEAM

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Keywords: contact problem, orthotropic plate, elastic inclusion, integro-differential equation, integral transformation, Riemann problem, asymptotic estimates

A piecewise-homogeneous elastic orthotropic plate, reinforced with a finite inclusion of the wedge-shaped, which meets the interface at a right angle and is loaded with tangential and normal forces is considered. By using methods of the theory of analytic function, the problem is reduced to singular integro-differential equations with fixed singularity. When the inclusion-beam has only bending stiffness and is loaded with normal forces, using an integral transformation a Riemann problem is obtained, the solution of which is presented in explicit form. The normal contact stresses along the contact line are determined and the behavior of the contact stresses in the neighborhood of singular points is established.

Doi: 10.54503/0002-3051-2022.75.1-2-184
Introduction

The solutions of static contact problems for different domains, reinforced with elastic thin inclusions and patches of variable stiffness were obtained, and the behavior of the contact stresses at the ends of the contact line has been investigated, depending on the geometrical and physical parameters of these thin-walled elements [1-10]. The first fundamental problem for a piecewise-homogeneous plane was solved, when a crack of finite length arrives at the interface of two bodies at the right angle [11], and also a similar problem for a piecewise-homogeneous plane when acted upon by symmetrical normal stresses at the crack sides [12, 13], as well as the contact problems for piecewise-homogeneous planes with a semi-infinite and finite inclusion [14].

Problem statement and its solution

Suppose an elastic body \( S \) occupies the plane of a complex variable \( z = x + iy \), which contains an elastic patch along the segment \( l_1 = (0,1) \) and consists of two half-planes of dissimilar materials

\[
S^{(1)} = \{ z \mid \text{Re} \ z > 0, z \notin [0,1] \}, \quad S^{(2)} = \{ z \mid \text{Re} \ z < 0 \}
\]

joined along the \( Oy \) axis. In particularly, we will consider a piecewise-homogeneous orthotropic plate in the condition of plane deformation, which consists of two half-planes of dissimilar materials and reinforced with a finite patch (inclusion) with modulus of elasticity \( E(x) \), thickness \( h(x) \) and Poisson’s coefficient \( \nu_l \). It is assumed that the horizontal and vertical stresses with intensity \( \tau_0(x) \) and \( p_0(x) \) acts on the patch along the \( Ox \) axis (the functions \( \tau_0(x) \) and \( p_0(x) \) are bounded functions on the finite interval). The patch in the vertical direction bends like a beam (has a finite bending stiffness) and besides in the horizontal direction the patch compressed or stretched like rod being in uniaxial stress state.

The contact between the plate and patch is realized by a thin glue layer with width \( h_0 \) and Lame’s constants \( \lambda_0, \mu_0 \). The contact conditions has the form [15]

\[
u_i(x) - u^{(i)}(x,0) = k_0 \tau(x), \quad v_i(x) - v^{(i)}(x,0) = m_0 p(x), \quad 0 < x < 1 \tag{1.1}
\]

where \( u^{(i)}(x,y), v^{(i)}(x,y) \) are displacement components of the plate points and \( u_i(x), v_i(x) \) displacements of the patch points along the \( Ox \) axis, \( k_0 = h_0/\mu_0 \) and \( m_0 = h_0/(\lambda_0 + 2\mu_0) \).

We have to define the law of distribution of tangential \( \tau(x) \) and normal \( p(x) \) contact stresses on the line of contact, the asymptotic behavior of these stresses at the ends of the patch.

According to the equilibrium equation of patch elements and Hooke's law we have:
\[ \frac{du_1(x)}{dx} = \frac{1}{E(x)} \int_0^x [\tau(t) - \tau_0(t)] dt, \]

\[ \frac{d^2}{dx^2} D(x) \frac{d^2 v_1(x)}{dx^2} = p_0(x) - p(x), \quad 0 < x < 1 \] (1.2)

and the equilibrium equation of the patch has the form

\[ \int_0^1 [\tau(t) - \tau_0(t)] dt = 0, \quad \int_0^1 [p(t) - p_0(t)] dt = 0, \quad \int_0^1 [p(t) - p_0(t)] dt = 0, \] (1.3)

where \( E(x) = \frac{E_i(x)h_i(x)}{1 - v_i^2} \), \( D(x) = \frac{E_i(x)h_i^3(x)}{1 - v_i^2} \).

At the interface of the two materials we have the continuity conditions

\[ \sigma_x^{(1)} = \sigma_x^{(2)}, \quad \tau_{xy}^{(1)} = \tau_{xy}^{(2)}, \quad u^{(1)} = u^{(2)}, \quad v^{(1)} = v^{(2)} \] (1.4)

where \( \sigma_x^{(k)}, \tau_{xy}^{(k)} \) are the stress components and \( u^{(k)}, v^{(k)} \) are the displacement components.

The boundary conditions of the components of the stress and displacement fields in the half-plane \( S^{(1)} \) have the form

\[ \sigma_y^{(1)} - \sigma_y^{(2)} = p(x), \quad \tau_{xy}^{(1)} - \tau_{xy}^{(2)} = \tau(x), \quad u^{(1)} = u^{(2)}, \quad v^{(1)} = v^{(2)} \] (1.5)

Using Lekhnitskii’s formulae [16] the components of stress and displacement are represented in the form

\[ \sigma_x^{(k)} = -2 \text{Re} \left[ \beta_k^2 \Phi_k(z_k) + \gamma_k^2 \psi_k(z_k) \right] \]

\[ \sigma_y^{(k)} = 2 \text{Re} \left[ \Phi_k(z_k) + \psi_k(z_k) \right] \]

\[ \tau_{xy}^{(k)} = 2 \text{Im} \left[ \beta_k \Phi_k(z_k) + \gamma_k \psi_k(z_k) \right] \] (1.6)

\[ u^{(k)} = 2 \text{Re} \left[ \rho_k \phi_k(z_k) + \rho_k \psi_k(z_k) \right] \]

\[ v^{(k)} = -2 \text{Im} \left[ \beta_k \psi_k(z_k) + \gamma_k \psi_k(z_k) \right] \]

\[ z_k = x + i \beta_k y, \quad \zeta_k = x + i \gamma_k y, \quad \Phi_k(z_k) = \phi_k'(z_k), \quad \Psi_k(z_k) = \psi_k'(z_k), \quad k = 1, 2 \]

Here \( \pm i \beta_k, \pm i \gamma_k \) are the roots of the characteristic equation

\[ \mu^4 + \left( \frac{E_k}{G_k} - 2v_k \right) \mu^2 + \frac{E_k}{E_k^*} = 0, \quad (\beta_k > \gamma_k), \]
\((E_k, E'_k)\) are the Young's modulus with respect to the principal \((Ox, Oy)\) direction respectively, \(G_k\) are the shear modulus, \(\nu_k\) are Poisson’s ratios of the plane materials, respectively.

The problem with conditions (1.1)-(1.5) reduced to finding the functions \(\Phi_k(z_k), \Psi_k(\xi_k), (k = 1, 2)\) which are holomorphic in the regions \(S^{(k)}\) respectively, and satisfies the following boundary conditions:

\[
\begin{align*}
2 \text{Re}[\Phi_k^+(x) - \Phi_k^-(x)] + \text{Im}[\gamma_1 \Psi'_k^+(x) - \Psi'_k^-(x)] &= p(x) \\
2 \text{Im}[\beta_1 \Phi'_k^+(x) - \Phi'_k^-(x)] + \gamma_1 \text{Re}[\Psi'_k^+(x) - \Psi'_k^-(x)] &= \tau(x) \\
\text{Re}[\rho_k \Phi'_k^+(x) - \Phi'_k^-(x)] + r'_1 \text{Re}[\Psi'_k^+(x) - \Psi'_k^-(x)] &= 0 \\
\text{Im}[\beta_1 r_1 (\Phi'_k^+(x) - \Phi'_k^-(x)) + \gamma_1 \rho_k (\Psi'_k^+(x) - \Psi'_k^-(x))] &= 0 \\
\text{Re}[\beta_1 \Phi_1(t_1) + \gamma_2 \Psi_1(t_1)] &= \text{Re}[\beta_2 \Phi_2(t_2) + \gamma_2 \Psi_2(t_2)] \\
\text{Im}[\beta_1 \Phi_1(t_1) + \gamma_2 \Psi_1(t_1)] &= \text{Im}[\beta_2 \Phi_2(t_2) + \gamma_2 \Psi_2(t_2)] \\
\text{Re}[\beta_1 r_1 \Phi_1(t_1) + \gamma_2 \rho \Psi_1(t_1)] &= \text{Re}[\beta_2 r_2 \Phi_2(t_2) + \gamma_2 \rho \Psi_2(t_2)]
\end{align*}
\]

(1.7)

where \(t_k = i\beta_k y, \sigma_k = i\gamma_k y, \rho_k = -\frac{\beta_k^2 + \nu_k}{E_k}, r_k = -\frac{\gamma_k^2 + \nu_k}{E_k}, k = 1, 2\)

System (1.7) has the unique solution

\[
\begin{align*}
\Phi_k^+(x) - \Phi_k^-(x) &= \frac{-r_1 \beta_k p(x) + i \rho_k \tau(x)}{2 \beta_k (\rho_1 - r_1)} \\
\Psi_k^+(x) - \Psi_k^-(x) &= \frac{\rho_k \gamma_1 p(x) - i r_1 \tau(x)}{2 \gamma_1 (\rho_1 - r_1)},
\end{align*}
\]

(1.9)

In view of the fact that \(\tau(x) = 0, \ p(x) = 0\) when \(x > 1\), the general solution of problem (1.9) can be represented in the form:

\[
\begin{align*}
\Phi_1(z_1) &= \frac{ir_1}{4\pi(\rho_1 - r_1)} \int_0^1 \frac{N_1(t)dt}{t - z_1} + w_1(z_1) + w_1(0(z_1)), \\
\Psi_1(\xi_1) &= -\frac{ip_1}{4\pi(\rho_1 - r_1)} \int_0^1 \frac{N_1(t)dt}{t - \xi_1} + w_2(\xi_1) - w_2(0(\xi_1)),
\end{align*}
\]

(1.10)

\[
\begin{align*}
N_1(t) &= p(t) - i \frac{\rho_1}{\rho_1 r_1} \tau(t), \\
N_2(t) &= p(t) - i \frac{r_1}{\rho_1 r_1} \tau(t).
\end{align*}
\]
where \( w_1(z_i) \) and \( w_2(\zeta_i) \) are unknown analytic functions in the half-planes \( \text{Re} \, z_i > 0, \text{Re} \, \zeta_i > 0 \) respectively, which will be defined by using the conditions (1.8).

We will now introduce the boundary values of functions \( \Phi_i(z_i) \) and \( \Psi_i'(\zeta_i) \), expressed by formulae (1.10), into equalities (1.8) and multiply expressions obtained by 

\[
\frac{1}{2\pi i} \frac{dt}{t-z},
\]

we integrate along the imaginary axis and use the fact that if \( \Phi(z) \) is a holomorphic function in the half-plane \( \text{Im} \, z > 0 (\text{Im} \, z < 0) \), then \( \overline{\Phi(iy)} \) is the boundary value of the function \( \Phi(-\overline{z}) \), holomorphic in the half-plane \( \text{Im} \, z < 0 (\text{Im} \, z > 0) \). As a result, using Cauchy’s theorem and formula, we obtain the following system

\[
\begin{align*}
\beta_1^2 w_1(\beta_1 z) + \gamma_1^2 w_2(\gamma_1 z) - \beta_2^2 \Phi_2(-\beta_2 \overline{z}) - \gamma_2^2 \Psi_2(-\gamma_2 \overline{z}) &= \\
= -ir \beta_1^2 w_0(-\overline{\beta_1 z}) + i r \gamma_1^2 w_0(-\overline{\gamma_1 z}) \\
\beta_1 w_1(\beta_1 z) + \gamma_1 w_2(\gamma_1 z) + \beta_2 \Phi_2(-\beta_2 \overline{z}) + \gamma_2 \Psi_2(-\gamma_2 \overline{z}) &= \\
= ir \beta_1 w_0(-\overline{\beta_1 z}) - ir \gamma_1 w_0(-\overline{\gamma_1 z}) \\
\rho \beta_1 w_1(\beta_1 z) + r \gamma_1 w_2(\gamma_1 z) + \rho \beta_2 \overline{\Phi_2(-\beta_2 \overline{z})} + \gamma_2 \overline{\Psi_2(-\gamma_2 \overline{z})} &= \\
= ir \rho \beta_1 w_0(-\overline{\beta_1 z}) - ir \rho \gamma_1 w_0(-\overline{\gamma_1 z}) \\
\beta_1^2 r_1 w_1(\beta_1 z) + \gamma_1^2 \rho w_2(\gamma_1 z) - \beta_2^2 r_2 \overline{\Phi_2(-\beta_2 \overline{z})} - \gamma_2^2 \rho \overline{\Psi_2(-\gamma_2 \overline{z})} &= \\
= -ir \beta_1^2 w_0(-\overline{\beta_1 z}) + i r \gamma_1^2 w_0(-\overline{\gamma_1 z})
\end{align*}
\]

Solving this system for functions \( w_1(\beta_1 z) \) and \( w_2(\gamma_1 z) \), and replacing \( z \) by \( \frac{z}{\beta_1} \)

and \( \frac{\zeta}{\gamma_1} \) respectively, one obtains

\[
\begin{align*}
w_1(z_i) &= \frac{i I_1}{\Delta} w_0(-\overline{z_i}) + \frac{i I_2}{\Delta} w_0(-\overline{\gamma_1 z_i}) \\
w_2(\zeta_i) &= \frac{i I^*_1}{\Delta} w_0(-\frac{\beta_1}{\gamma_1} \overline{\zeta_i}) + \frac{i I^*_2}{\Delta} w_0(-\overline{\zeta_i})
\end{align*}
\]

(1.11)

For functions \( \Phi_2(-\beta_2 z) \) and \( \Psi_2(-\gamma_2 z) \) with this notation \(-\beta_2 z = z_2, -\gamma_2 z = \zeta_2\), we have
\[ \Phi_2(z_2) = -\frac{iI_z}{\Delta} w_0(\frac{\beta_1}{\beta_2}, z_2) - \frac{iI_z}{\Delta} w_0(\frac{\gamma_1}{\beta_2}, z_2), \]

\[ \Psi_2(z_2) = -\frac{iI_z^*}{\Delta} w_0(\frac{\beta_1}{\gamma_2}, z_2) - \frac{iI_z^*}{\Delta} w_0(\frac{\gamma_1}{\gamma_2}, z_2), \]

where

\[ I_1 = -\Delta_1 r_1 \beta_1^2 + \Delta_2 r_1 \beta_1 + \Delta_3 r_1 \rho_1 \beta_1 - \Delta_4 \beta_1^2 \tau_1^2, \]

\[ I_2 = \Delta_1 r_1 \gamma_1^2 - \Delta_2 r_1 \gamma_1 - \Delta_3 r_1 \rho_1 \gamma_1 + \Delta_4 \gamma_1^2 \tau_1^2, \]

\[ I_3 = -\Delta_1 r_1 \beta_2^2 + \Delta_2 r_1 \beta_2 + \Delta_3 r_1 \rho_1 \beta_2 - \Delta_4 \beta_2^2 \tau_2^2, \]

\[ I_4 = \Delta_1 r_1 \gamma_2^2 - \Delta_2 r_1 \gamma_2 - \Delta_3 r_1 \rho_1 \gamma_2 + \Delta_4 \gamma_2^2 \tau_2^2 \]

\[ \Delta = \begin{vmatrix} \beta_1^2 & \gamma_1^2 & -\beta_2^2 & -\gamma_2^2 \\ \beta_1 & \gamma_1 & \beta_2 & \gamma_2 \\ \rho_1 \beta_1 & \tau_1 \gamma_1 & \rho_2 \beta_2 & \tau_2 \gamma_2 \\ \beta_1^* \gamma_1 & \beta_2^* \gamma_2 & -\beta_2^* \gamma_2 & -\gamma_2^2 \rho_2 \end{vmatrix}, \]

\[ \Delta_{ij} (i, j = 1, 2, 3, 4) \] are the cofactors of the corresponding matrix elements.

Boundary conditions (1.2) are equivalent to the relations:

\[ \frac{1}{E(x)} \int_0^x \left[ \tau_1 (t) - \tau_1^0 (t) \right] dt \left[ p_1 \Phi_1(x) + p_1 \Phi_1^*(x) + r_1 \Psi_1(x) + r_1 \Psi_1^*(x) \right] = k_0 \tau'(x) \]

\[ \frac{1}{D(x)} \int_0^x \left[ p_1^0 (\tau) - p_1 (\tau) \right] d\tau = -i \frac{d}{dx} \left[ \beta_1 \Phi_1(x) - \Phi_1^*(x) + \gamma_1 \rho_1 \left( \Psi_1(x) - \Psi_1^*(x) \right) \right] = m_0 p_1^* (x) \]

Substituting expressions (1.10) and (1.11) into (1.12) we obtain the integro-differential equations on the interval \( 0 < x < 1 \)
\[ \frac{\psi(x)}{E(x)} - \frac{1}{2\pi} \int_0^1 Q(t, x) \psi'(t) dt = k_0 \psi''(x) = f_1(x), \]  
(1.13)

\[ \frac{\varphi(x)}{D(x)} + \frac{1}{2\pi} \frac{d}{dx} \int_0^1 R(t, x) \varphi^*(t) dt + m_0 \varphi^{iv}(x) = f_2(x), \]  
(1.14)

\[ \psi(1) = 0, \quad \varphi(1) = 0, \quad \varphi'(1) = 0 \]

where

\[ Q(t, x) = \frac{\lambda_1}{t-x} + \frac{\lambda_2}{t+x} + \frac{\lambda_3}{\beta_1 t + \gamma_1 x} + \frac{\lambda_4}{\gamma_1 t + \beta_1 x}, \]

\[ R(t, x) = \frac{k_1}{t-x} + \frac{k_2}{t+x} + \frac{k_3}{\beta_2 t + \gamma_2 x} + \frac{k_4}{\gamma_2 t + \beta_2 x}, \]

\[ \psi(x) = \int_0^x [\tau(t) - \tau_0(t)] dt, \quad \varphi(x) = \int_0^x \int_0^t [p_0(\tau) - p(\tau)] d\tau, \]

\[ f_1(x) = \frac{1}{2\pi} \int_0^1 Q(t, x) \tau_0(t) dt + k_0 \tau'_0(x), \]

\[ f_2(x) = m_0 \tau_0^*(x) + \frac{1}{2\pi} \frac{d}{dx} \int_0^1 R(t, x) p_0(t) dt \]

\[ \lambda_1 = \frac{\rho_1^2 \gamma_1 - r_1^2 \beta_1}{(\rho_1 - r_1) \beta_1}, \quad \lambda_2 = \frac{\rho_2^2 \gamma_1 I_1 + r_2^2 \beta_1 I_2^*}{\Delta \beta_1 \gamma_1 (\rho_1 - r_1)}, \quad \lambda_3 = \frac{-I_2 p_1^2}{\Delta \rho_1 (\rho_1 - r_1)}, \quad \lambda_4 = \frac{-I_2 r_1^2}{\Delta \rho_1 (\rho_1 - r_1)}, \]

\[ k_1 = \frac{\beta_1 r_1^2 + \gamma_1 \rho_1^2}{\rho_1 - r_1}, \quad k_2 = \frac{\beta_2 r_1 I_1 + \gamma_1 \rho_1 I_2^*}{\Delta (\rho_1 - r_1)}, \quad k_3 = \frac{\beta_1 \rho_1 I_2^*}{\Delta (\rho_1 - r_1)}, \quad k_4 = \frac{\gamma_1 \rho_1 I_2^*}{\Delta (\rho_1 - r_1)}. \]

**Exact solution of equation (1.14)**

Under the condition, when the inclusion-beam is loaded only with normal forces and bending stiffness of the inclusion varies linearly, i.e. \( D(x) = d_0 x^3 \), \( m_0(x) = m_0 x \), the equation (1.14) and the corresponding boundary conditions take the form

\[ \frac{\varphi(x)}{D(x)} - \frac{1}{2\pi} \frac{d}{dx} \int_0^1 R(t, x) \varphi^*(t) dt + [m_0(x) \varphi^*(x)]' = f_2(x), \quad 0 < x < 1 \]  
(2.1)

\[ \varphi(1) = 0, \quad \varphi'(1) = 0 \]
\[ f_2(x) = - \frac{d}{dx} \int_0^1 \frac{p_0(t)dt}{t-x} + m_0(x)p_0''(x), \]

The solution of equation (2.1) is sought in the class of functions \([17]\)
\[ \varphi, \varphi', \varphi'', \varphi''' \in H((0,1)), \quad \varphi'''' \in H((0,1)). \]

The change of variables \( x = e^\xi, \quad t = e^\xi \) in equation (2.1) gives

\[ \frac{\varphi_0(\xi)}{d_0 e^{2\xi}} - \frac{1}{2\pi e^\xi d\xi} \int_{-\infty}^0 e^{-2\xi} \left( 1, e^{\xi-\xi} \right) \left[ \varphi_0''(\xi) - \varphi_0'(\xi) \right] d\xi + \]
\[ + m_0 e^{-2\xi} \left[ \varphi_0''''(\xi) - 4\varphi_0''(\xi) + 5\varphi_0'(\xi) - 2\varphi_0'(\xi) \right] = f_2(e^\xi), \quad -\infty < x < 0, \]

where \( \varphi_0(\xi) = \varphi(e^\xi) \).

Subjecting both part of this equation to generalized Fourier transform \([18]\) we obtain the following Riemann boundary value condition:

\[ \Phi^-(s)G(s) = \Psi^+(s) + F(s), \quad \left| s \right| < \infty \quad (2.2) \]

where

\[ G(s) = 1 + \frac{d_0}{2} \left[ k_s \coth \pi s + k_\gamma \frac{s}{\pi s} + k_\lambda \frac{e^{i\lambda}}{\gamma s} + k_\nu \frac{s}{\nu s} \right] (s-i) + \]
\[ + \lambda_s^4 (s^4 - 4is^3 - 5s^2 + 2is), \quad \mu = \ln \frac{\gamma_1}{\gamma}, \quad \lambda_2^4 = m_0 d_0 \]

\[ \Phi^-(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \varphi(e^\xi)e^{is\xi}d\xi, \quad F(s) = \frac{2}{\pi} \int_{-\infty}^0 e^{\xi} f_2(e^\xi)e^{is\xi}d\xi, \]

\[ \Psi^+(s) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \psi(\xi)e^{is\xi}d\xi \]

\[ \psi(y) = \begin{cases} 
0, & y < 0 \\
e^{2y} \frac{d}{dy} \int_{-\infty}^y e^{-2s} \left( 1, e^{s-y} \right) \left[ \varphi_0''(s) - \varphi_0'(s) \right] ds, & y > 0 
\end{cases} \]

The condition (2.2) can be represented as

\[ \Phi^-(s)(1 + \sqrt{i\lambda_2}s)(1 + i\sqrt{i\lambda_2}s)G_0(s) = \frac{\Psi^+(s)}{(1 - \sqrt{i\lambda_2}s)(1 - i\sqrt{i\lambda_2}s)} + H(s) \quad (2.3) \]
where

\[
G(s) = \frac{G(s)}{1 + \lambda_2^2 s^2} (1 + i\lambda_2^2 s^2)(1 - i\lambda_2^2 s^2) \\
= G_0(s)(1 - \sqrt{i\lambda_2 s})(1 - i\sqrt{i\lambda_2 s})(1 + \sqrt{i\lambda_2 s})(1 + i\sqrt{i\lambda_2 s})
\]

\[
H(s) = \frac{F(s)}{(1 - \sqrt{i\lambda_2 s})(1 - i\sqrt{i\lambda_2 s})}, \quad G_0(s) = \frac{G(s)}{1 + \lambda_2^2 s^4}.
\]

By virtue of functions \( \Phi^-(s) \) and \( \Psi^+(s) \) definition, they will be boundary values of the functions which are holomorphic in the upper and lower half-planes, respectively.

The problem can be formulated as follows: it is required to determine the function \( \Psi^+(z) \), holomorphic in the half-plane \( \text{Im} \ z > 0 \) and which vanishes at infinity, and the function \( \Phi^- (z) \), holomorphic in the half-plane \( \text{Im} \ z < 1 \), (with the exception of a finite number of zeros of function \( G(z) \)) which vanishes at infinity and are continuous on the real axis by condition (2.3).

Since \( \text{Re} \ G_0(s) > 0 \) and \( G_0(\infty) = G_0(-\infty) = 1 \), we have \( \text{Ind} G_0(s) = 0 \).

The solution of this problem has the form [17]

\[
\Phi^- (z) = \left\{ \begin{array}{l}
\tilde{X}(z) (1 + \sqrt{i\lambda_2 s})(1 + i\sqrt{i\lambda_2 s}) \\
\tilde{X}(z)(1 - \sqrt{i\lambda_2 s})(1 - i\sqrt{i\lambda_2 s}), \quad \text{Im} \ z > 0
\end{array} \right.
\]

\[
\Psi^+ (z) = \Phi^- (z) + F(z)G^{-1}(z), \quad 0 < \text{Im} \ z < 1
\]  

(2.4)

where

\[
\tilde{X}(z) = X(z) \left\{ -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{H(t) dt}{X^\prime(t)(t - z)} \right\}, \quad X(z) = \exp \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln G_0(t) dt}{t - z} \right\}.
\]

(here the integral should be understood in the sense of the Cauchy principal value).

Using the formula \( \varphi^a(x) = \frac{\varphi_0^a(\ln x) - \varphi_0^\prime(\ln x)}{x^2} \) and applying the inverse Fourier transformation

\[
\varphi_0^\prime(\ln x) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} s\Phi^-(s)e^{-is\ln x} ds, \quad \varphi_0^a(\ln x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} s^2\Phi^-(s)e^{-is\ln x} ds
\]
we will investigate the behavior of the function \( p_0(x) - p(x) = \psi^*(x) \) in the neighborhood of the points \( z = 0 \) and \( z = 1 \).

We obtain by an inverse transformation: \( p_0(x) - p(x) = O(1), \quad x \to 1^- \).

The poles of the function \( \Phi^-(z) \) in the domain \( D_0 = \{z : 0 < \text{Im} \, z < 1\} \) may be zeros of the function \( G(z) \). It can be shown that the function \( G(z) \) has no zeros in the strip \( 0 < \text{Im} \, z < 2 \). Then, applying Cauchy’s theorem to the functions \( e^{-\frac{\pi}{2}iz} \Phi^- (z) \), \( e^{-\frac{\pi}{2}iz^2} \Phi^- (z) \) we obtain the following estimate

\[
p_0(x) - p(x) = O(x_0^{-y_0-2+iy_0}), \quad x \to 0^+, \quad y_0 > 2
\]

where \( z_0 = x_0 + iy_0 \) is zero of the function \( G(z) \) with a minimal imaginary part and with \( x_0 \neq 0 \), consequently we have oscillating stress singularities.

References


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Received 26.02.2022