Известия НАН Армении, Математика, том 57, н. 2, 2022, стр. 70 – 80.

SMOOTH FUNCTIONS AND GENERAL FOURIER COEFFICIENTS

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https://doi.org/10.54503/0002-3043-2022.57.2-70-80

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Abstract. As is known, the Fourier series of differentiable functions for classical orthonormal systems (trigonometric, Haar, Walsh, . . .) are absolutely convergent. However, for general orthonormal systems (ONS) this fact does not hold. In the present paper, we consider some specific properties of special series of Fourier coefficients of differentiable functions with respect to the general ONS. The obtained results demonstrate that the properties of the general ONS and of the subsequence of this system are essentially different. Here we have shown that the received results are best possible.

MSC2020 numbers: 42C10.

Keywords: Fourier coefficients; function of bounded variation; absolutely continuous function; general orthonormal system.

1. Auxiliary notations and theorems

Let (φ_n) be an ONS on $[0;1]$. Suppose that $f \in L_2$, then the Fourier coefficients of the function f are defined as follows:

$$C_n(f) = \int_0^1 f(x)\varphi_n(x)\,dx, \quad n = 1, 2, \dots . \tag{1.1}$$

We denote

$$D_n(a) = \max_{1\le i\le n} \left| \int_0^{\frac{i}{n}} B_n(a, x)\,dx \right|, \tag{1.2}$$

where

$$B_n(a; x) = \sum_{k=1}^{n} a_k k^{\gamma} \int_0^x \varphi_k(u)\,du.$$

Also,

$$E_n(a, x) = \sum_{k=1}^{n} a_k k^{\gamma} \varphi_k(x)$$

and (a_k) is some sequence of real numbers.

Assume that $(0 < \gamma < 1)$

$$H_n(a) = \left(\sum_{k=1}^{n} a_k^2 k^{\gamma} \right)^{\frac{1}{2}}.$$

The bounded sequence we denote by (r_n), $r_n = O(1)$.

Lemma 1.1. *For any* $(a_n) \in \ell_2$ *and for* $h(x) = 1$ $x \in [0,1]$ *there holds*

$$\left| \int_0^1 E_n(a,x)\,dx \right| = O(1)\, H_n(a) H_n(C(h)),$$

where $H_n(C(h)) = H_n(a)$ *and* $C_k(h) = \int_0^1 h(x)\varphi_k(x)\,dx = a_k$ $(a = (a_k)$ *and* $C = (C_k(f))$.

Proof. According to the Cauchy inequality, we have (see (1.1))

$$\begin{aligned} \left| \int_0^1 E_n(a,x)\,dx \right| &= \left| \sum_{k=1}^n a_k k^\gamma \int_0^1 h(u)\varphi_k(u)\,du \right| \\ &\le \left(\sum_{k=1}^n a_k^2 k^\gamma \right)^{\frac{1}{2}} \left(\sum_{k=1}^n C_k^2(h) k^\gamma \right)^{\frac{1}{2}} = O(1) H_n(a) H_n(C(h)). \end{aligned}$$

Lemma 1.1 is proved. □

Lemma 1.2. *Let* $g(x) = x$ *for* $x \in [0,1]$ *and* $(a_n) \in \ell_2$*, then*

$$\left| \int_0^1 B_n(a,x)\,dx \right| = O(1) H_n(a) \big(H_n(C(h)) + H_n(C(g)) \big).$$

Proof. Integrating by parts we obtain

$$\begin{aligned} \int_0^1 B_n(a,x)\,dx &= \int_0^1 E_n(a,x)\,dx - \sum_{k=1}^n a_k k^\gamma \int_0^1 x\varphi_k(x)\,dx \\ &= \int_0^1 E_n(a,x)\,dx - \sum_{k=1}^n a_k k^\gamma C_k(g). \end{aligned}$$

Hence, using Hölder's inequality, from Lemma 1.1 it follows

$$\left| \int_0^1 B_n(a,x)\,dx \right| = O(1) H_n(a) H_n(C(h)) + O(1) H_n(a) H_n(C(g)).$$

Lemma 1.2 is proved. □

Lemma 1.3. *If* $(a_k) \in \ell_2$, $\gamma < 1$*, then for any* $i = 1, 2, \ldots, n$

$$\int_{\frac{i-1}{n}}^{\frac{i}{n}} |B_n(a,x)|\,dx = O(1).$$

Proof. By the Cauchy inequality

$$\begin{aligned} \left| \int_{\frac{i-1}{n}}^{\frac{i}{n}} B_n(a,x)\,dx \right| &= \left| \int_{\frac{i-1}{n}}^{\frac{i}{n}} \sum_{k=1}^n a_k k^\gamma \int_0^x \varphi_k(u)\,du \right| \\ &\le \frac{1}{\sqrt{n}} \left(\int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\sum_{k=1}^n a_k k^\gamma \int_0^x \varphi_k(u)\,du \right)^2 \right)^{\frac{1}{2}} \\ &\le \frac{1}{\sqrt{n}} \left(\sum_{k=1}^n a_k^2 k^{2\gamma} \right)^{\frac{1}{2}} \left(\int_{\frac{i-1}{n}}^{\frac{i}{n}} \sum_{k=1}^n \left(\int_0^x \varphi_k(u)\,du \right)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Then, according to the Bessel inequality, we have

$$\sum_{k=1}^{\infty}\left(\int_0^x \varphi_k(u)\,du\right)^2 \le 1.$$

Since $\gamma < 1$ and $\frac{1}{n}n^\gamma < 1$, we get

$$\left|\int_{\frac{i-1}{n}}^{\frac{i}{n}} B_n(a,x)\,dx\right| \le \frac{1}{\sqrt{n}}n^\gamma\left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}}\frac{1}{\sqrt{n}} = O(1).$$

Lemma 1.3 is proved. □

Theorem 1.1 (see [1]). *Let $f, F \in L_2$. Then*

$$\int_0^1 f(x)F(x)\,dx = n\sum_{i=1}^{n-1}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\left(f(x) - f\left(x+\frac{1}{n}\right)\right)dx\int_0^{\frac{i}{n}} F(x)\,dx \tag{1.3}$$
$$+\, n\sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{i-1}{n}}^{\frac{i}{n}}(f(x)-f(t))dt\;F(x)\,dx + n\int_{1-\frac{1}{n}}^{1} f(x)\,dx\int_0^1 F(x)\,dx.$$

By V we denote the class of functions of bounded variation and by $V(f)$ the finite variation of function f on $[0,1]$. Let C_V be a class of functions f for which $f'(x) = \frac{d}{dx}f(x) \in V$.

By A we denote the class of absolutely continuous functions. A is a Banach space with the norm

$$\|f\|_A = \|f\|_C + \int_0^1 |f'(x)|\,dx.$$

2. The main problem

Suppose that $f \in C_V$ is an arbitrary function and (φ_n) is trigonometric [2, Ch. 4], Haar [3, Ch. 1] or Walsh [3, Ch. 1] system, then it is evident that if $0 < \gamma < 1$,

$$\sum_{k=1}^{\infty} C_k^2(f)k^\gamma = O(1)\sum_{k=1}^{\infty} k^{-2}k^\gamma < +\infty.$$

There arises the question: is the series

$$\sum_{k=1}^{\infty} C_k^2(f)k^\gamma$$

convergent for any $f \in C_V$ and for arbitrary ONS when $0 < \gamma < 1$?

It is known (see [4]) that if $f \in L_2$ is an arbitrary function ($f \nsim 0$) and $(a_k) \in \ell_2$ is an arbitrary sequence of numbers, then there exists an ONS (φ_n) such that

$$C_n(f) = da_n, \quad n = 1, 2, \dots \quad (d \ne 0 \text{ depends only on } f \text{ and } (a_n)).$$

Assume that $g(x) = 1$ for $x \in [0;1]$ and let

$$a_n = \frac{1}{\sqrt{n}\log(n+1)}.$$

Then, since $(a_n) \in \ell_2$ as it was noted above, there exists an ONS (φ_n) such that

$$C_n(g) = da_n, \quad n = 1, 2, \dots .$$

Hence

$$\sum_{k=1}^{\infty} C_k^2(g) k^{\gamma} = d^2 \sum_{k=1}^{\infty} \frac{1}{k \log^2(k+1)} k^{\gamma} = +\infty,$$

though in this case $g \in C_V$.

The similar problems are considered in the papers [5]-[8].

3. The main results

Theorem 3.1. *Let (φ_n) be an ONS on $[0;1]$ such that $H_n(C(h)) = O(1)$ and $H_n(C(g)) = O(1)$ (see Lemmas 1.1 and 1.2). If for arbitrary $(a_n) \in \ell_2$ (see (1.2))*

$$D_n(a) = O(1) H_n(a), \tag{3.1}$$

then for any $f \in C_V$, $0 < \gamma < 1$, there holds

$$\sum_{k=1}^{n} C_k^2(f) k^{\gamma} < +\infty.$$

Proof. For arbitrary $f \in L_2(0,1)$,

$$\sum_{k=1}^{n} C_k^2(f) k^{\gamma} = \sum_{k=1}^{n} C_k(f) k^{\gamma} \int_0^1 f(x) \varphi_k(x)\, dx = \int_0^1 f(x) \sum_{k=1}^{n} C_k(f) k^{\gamma} \varphi_k(x)\, dx = \int_0^1 f(x) E_n(C,x)\, dx, \tag{3.2}$$

where $E_n(C,x) = E_n(a,x)$ when $C_k(f) = a_k$, $k = 1, 2, \dots$.

In (1.3) we substitute $F(x) = B_n(C;x)$ and $f(x) = f'(x)$:

$$\begin{aligned} \int_0^1 f'(x) B_n(C,x)\, dx &= n \sum_{i=1}^{n-1} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \Big(f'(x) - f'\Big(x + \frac{1}{n}\Big)\Big)\, dx \int_0^{\frac{i}{n}} B_n(C,x)\, dx \\ &+ n \sum_{i=1}^{n-1} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \big(f'(x) - f'(t)\big) dt\, B_n(C,x)\, dx \\ &+ n \int_{1-\frac{1}{n}}^{1} f'(x)\, dx \int_0^1 B_n(C,x)\, dx = P_1 + P_2 + P_3. \end{aligned} \tag{3.3}$$

By conditions (3.1) and $f \in C_V$ we get ($\Delta_{in} = [\frac{i-1}{n}, \frac{i}{n}]$)

$$\begin{aligned} |P_1| &= n O\Big(\frac{1}{n}\Big) \sum_{i=1}^{n-1} \sup_{x \in \Delta_{in}} \Big| f'(x) - f'\Big(x + \frac{1}{n}\Big)\Big| \Big| \int_0^{\frac{i}{n}} B_n(C,x)\, dx \Big| \\ &= O(1) V(f') D_n(C) = O(1) H_n(C). \end{aligned} \tag{3.4}$$

According to Lemma 1.3 ($0<\gamma<1$),

(3.5)
$$|P_2| = nO\Big(\frac{1}{n}\Big)\sum_{i=1}^{n}\max_{x,t\in\Delta_{in}}|f'(x)-f'(t)|\int_{\frac{i-1}{n}}^{\frac{i}{n}}|B_n(C,x)|\,dx = O(1)V(f') = O(1).$$

Next, Lemma 1.2 and conditions of Theorem 3.1 imply

(3.6)
$$|P_3| = \Big|\int_0^1 B_n(a,x)\,dx\Big| = O(1)H_n(C)\big(H_n(C(h))+H_n(C(g))\big) = O(1)H_n(C) = O(1).$$

Taking into account (3.4), (3.5) and (3.6) in (3.3) we get

$$\Big|\int_0^1 f'(x)B_n(C,x)\,dx\Big| = O(1)H_n(C) + O(1). \tag{3.7}$$

Using (3.2) and integration by parts we have

(3.8)
$$\sum_{k=1}^{n} C_k^2(f)k^\gamma = \int_0^1 f(x)E_n(C,x)\,dx = f(1)\int_0^1 E_n(C,x)\,dx - \int_0^1 f'(x)B_n(C,x)\,dx.$$

It can be easily verified that (see (3.8), (3.7) and Lemma 1.1)

$$\sum_{k=1}^{n} C_k^2(f)k^\gamma = O(1)H_n(C)H_n(C(h)) + O(1)H_n(C) + O(1)$$

$$= O(1) + O(1)\Big(\sum_{k=1}^{n} C_k^2(f)k^\gamma\Big)^{\frac{1}{2}}.$$

So

$$\Big(\sum_{k=1}^{n} C_k^2(f)k^\gamma\Big)^{\frac{1}{2}} = O(1).$$

Finally, for any $f\in C_V$,

$$\sum_{k=1}^{\infty} C_k^2(f)k^\gamma < +\infty.$$

Theorem 3.1 is proved. □

Theorem 3.2. *Let (φ_n) be an ONS on $[0;1]$. If for some $(b_n)\in\ell_2$*

$$\limsup_{n\to\infty}\frac{1}{H_n(b)}\,|D_n(b)| = +\infty, \tag{3.9}$$

then there exists a function $s\in C_V$ such that

$$\sum_{n=1}^{\infty} C_n^2(s)n^\gamma = +\infty.$$

Proof. First, we suppose that

$$\lim_{n\to\infty} H_n(C(h)) = +\infty \quad\text{or}\quad \lim_{n\to\infty} H_n(C(g)) = +\infty.$$

Since $h(x) = 1$ and $g(x) = x$, when $x \in [0,1]$, we conclude that

$$\lim_{n\to\infty} \sum_{k=1}^{n} C_k^2(h)k^\gamma = +\infty \quad \text{or} \quad \lim_{n\to\infty} \sum_{k=1}^{n} C_k^2(g)k^\gamma = +\infty.$$

In such a case Theorem 3.2 is proved.

Now we assume that

$$H_n(C(h)) = O(1) \quad \text{and} \quad H_n(C(g)) = O(1). \tag{3.10}$$

We have

$$D_n(a) = \max_{1\le i\le n} \left| \int_0^{\frac{i}{n}} B_n(a,x)\,dx \right| = \left| \int_0^{\frac{i_n}{n}} B_n(a,x)\,dx \right|, \text{ where } 1 \le i_n < n.$$

Here we must note that if $i_n = n$ and

$$\limsup_{n\to\infty} \frac{1}{H_n(b)} \left| \int_0^1 B_n(a,x)\,dx \right| = +\infty,$$

then according to Lemma 1.3

$$\left| \int_{1-\frac{1}{n}}^1 B_n(a,x)\,dx \right| = O(1)$$

and

$$\begin{aligned}\limsup_{n\to\infty} &\frac{1}{H_n(b)} \left| \int_0^{1-\frac{1}{n}} B_n(a,x)\,dx \right| \\ &= \limsup_{n\to\infty} \frac{1}{H_n(b)} \left| \int_0^1 B_n(a,x)\,dx \right| - \limsup_{n\to\infty} \frac{1}{H_n(b)} \left| \int_{1-\frac{1}{n}}^1 B_n(a,x)\,dx \right| = +\infty.\end{aligned}$$

We define the sequence of functions (f_n) as follows:

$$f_n(x) = \begin{cases} 0 & \text{when } x \in [0, \frac{i_n-2}{n}], \\ 1 & \text{when } x \in [\frac{i_n}{n}, 1], \\ \frac{nx-i_n+2}{2} & \text{when } x \in [\frac{i_n-2}{n}, \frac{i_n}{n}]. \end{cases} \tag{3.11}$$

In (3.3) we substitute $f' = f_n$, then

(3.12)

$$\begin{aligned}\int_0^1 f_n(x)B_n(b,x)\,dx &= n\sum_{i=1}^{n-1} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \Big(f_n(x) - f_n\Big(x+\frac{1}{n}\Big)\Big)\,dx \int_0^{\frac{i}{n}} B_n(b,x)\,dx \\ &+ n\sum_{i=1}^{n-1} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{i-1}{n}}^{\frac{i}{n}} (f_n(x) - f_n(t))dtB_n(b,x)\,dx \\ &+ n\int_{1-\frac{1}{n}}^1 f_n(x)\,dx \int_0^1 B_n(b,x)\,dx = S_1 + S_2 + S_3.\end{aligned}$$

By (3.11), since $|f_n(x) - f_n(t)| \le 1$ when $x, t \in [0,1]$ and $f_n(x) - f_n(t) = 0$ when $x, t \in [0, \frac{i_n-2}{n}]$ or $x, t \in [\frac{i_n}{n}, 1]$, using Lemma 1.1, we receive

$$|S_2| \le n\frac{1}{n} \int_{\frac{i_n-2}{n}}^{\frac{i_n}{n}} |B_n(b,x)|\,dx = O(1). \tag{3.13}$$

Next, taking into account Lemma 1.2 and (3.10), we get

$$\left|\int_0^1 B_n(b,x)\,dx\right| = O(1)H_n(b)H_n(C(h)) + O(1)H_n(b)H_n(C(g)) = O(1)H_n(b).$$

Hence it follows that

$$|S_3| \le n\int_{1-\frac{1}{n}}^1 |f_n(x)|\,dx\left|\int_0^1 B_n(b,x)\,dx\right| = O(1)H_n(b). \tag{3.14}$$

Taking into consideration (3.11) we have

a) $\displaystyle\int_{\frac{i_n-3}{n}}^{\frac{i_n-2}{n}} \Big(f_n(x) - f_n\Big(x+\frac{1}{n}\Big)\Big)\,dx = -\int_{\frac{i_n-2}{n}}^{\frac{i_n-1}{n}} \frac{nx-i_n+2}{2}\,dx = -\frac{1}{4n}\,;$

b) $\displaystyle\int_{\frac{i_n-2}{n}}^{\frac{i_n-1}{n}} \Big(f_n(x) - f_n\Big(x+\frac{1}{n}\Big)\Big)\,dx = -\frac{1}{2n};$

c) $\displaystyle\int_{\frac{i_n-1}{n}}^{\frac{i_n}{n}} \Big(f_n(x) - f_n\Big(x+\frac{1}{n}\Big)\Big)\,dx = \int_{\frac{i_n-1}{n}}^{\frac{i_n}{n}} \frac{nx-i_n+2}{2}\,dx - \frac{1}{n}$
$\displaystyle\qquad = \frac{3}{4n} - \frac{1}{n} = -\frac{1}{4n}\,;$

d) $\displaystyle\int_{\frac{i-1}{n}}^{\frac{i}{n}} \Big(f_n(x) - f_n\Big(x+\frac{1}{n}\Big)\Big)\,dx = 0$ when $i \le i_n - 3$ or $i \ge i_n + 1$.

Therefore, due to a)–d) we get

$$\begin{aligned}
|S_1| &= n\left|\sum_{i=i_n-2}^{i_n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} \Big(f_n(x) - f_n\Big(x+\frac{1}{n}\Big)\Big)\,dx\int_0^{\frac{i}{n}} B_n(b,x)\,dx\right| \\
&= n\left|\frac{1}{4n}\int_0^{\frac{i_n-2}{n}} B_n(b,x)\,dx + \frac{1}{2n}\int_0^{\frac{i_n-1}{n}} B_n(b,x)\,dx + \frac{1}{4n}\int_0^{\frac{i_n}{n}} B_n(b,x)\,dx\right| \\
&\ge \left|\int_0^{\frac{i_n}{n}} B_n(b,x)\,dx\right| - \frac{1}{4}\int_{\frac{i_n-2}{n}}^{\frac{i_n}{n}} |B_n(b,x)|\,dx - \frac{1}{2}\int_{\frac{i_n-1}{n}}^{\frac{i_n}{n}} |B_n(b,x)|\,dx.
\end{aligned}$$

Since (see Lemma 1.3)

$$\int_{\frac{i_n-2}{n}}^{\frac{i_n}{n}} |B_n(b,x)|\,dx = O(1) \quad\text{and}\quad \int_{\frac{i_n-1}{n}}^{\frac{i_n}{n}} |B_n(b,x)|\,dx = O(1),$$

we have

$$|S_1| \ge D_n(b) - O(1). \tag{3.15}$$

Hence from (3.12), because of (3.13), (3.14) and (3.15), it follows

$$\left|\int_0^1 f_n(x)B_n(b,x)\,dx\right| \ge D_n(b) - O(1).$$

From here and (3.9),

$$\limsup_{n\to\infty}\frac{1}{H_n(b)}\left|\int_0^1 f_n(x)B_n(b,x)\,dx\right| = +\infty. \tag{3.16}$$

It can be easily verified that

$$\Delta_n(f) = \frac{1}{H_n(b)} \int_0^1 f(x) B_n(b,x)\, dx, \quad n = 1, 2, \dots ,$$

is a sequence of linear and bounded functionals on A.

On the other hand,

$$||f_n||_A = ||f_n||_C + \int_0^1 |f_n'(x)|\, dx \le 2. \tag{3.17}$$

Since (3.16)

$$\limsup_{n\to\infty} |\Delta_n(f_n)| = +\infty$$

and (3.17), by virtue of Banach–Steinhaus Theorem there exists a function $u \in A$ such that

$$\limsup_{n\to\infty} \frac{1}{H_n(b)} \left| \int_0^1 u(x) B_n(b,x)\, dx \right| = +\infty. \tag{3.18}$$

We assume that

$$s(x) = \int_0^x u(t) dt.$$

It can be easily verified (see (3.8)) that

$$\sum_{k=1}^n C_k(s) b_k k^\gamma = \int_0^1 s(x) \sum_{k=1}^n b_k k^\gamma \varphi_k(x)\, dx = \int_0^1 s(x) E_n(b,x)\, dx$$
$$= s(1) \int_0^1 E_n(b,x)\, dx - \int_0^1 s'(x) B_n(b,x)\, dx.$$

From here, since $s'(x) = u(x)$, by virtue of Lemma 1.1 and (3.10) (see (3.18)), we get

$$\limsup_{n\to\infty} \frac{1}{H_n(b)} \left| \sum_{k=1}^n C_k(s) b_k k^\gamma \right| \ge \limsup_{n\to\infty} \frac{1}{H_n(b)} \left| \int_0^1 u(x) B_n(b,x)\, dx \right| - \limsup_{n\to\infty} \frac{|s(1)|}{H_n(b)} \left| \int_0^1 E_n(b,x)\, dx \right| = +\infty. \tag{3.19}$$

Now using the Cauchy inequality,

$$\left| \sum_{k=1}^n b_k k^\gamma C_k(s) \right| \le \left(\sum_{k=1}^n b_k^2 k^\gamma \right)^{\frac12} \left(\sum_{k=1}^n C_k^2(s) k^\gamma \right)^{\frac12} = H_n(b) \left(\sum_{k=1}^n C_k^2(s) k^\gamma \right)^{\frac12}.$$

Finally, due to (3.19) we get

$$\lim_{n\to\infty} \left(\sum_{k=1}^n C_k^2(s) k^\gamma \right)^{\frac12} = \limsup_{n\to\infty} \frac{1}{H_n(b)} \left| \sum_{k=1}^n b_k k^\gamma C_k(s) \right| = +\infty.$$

Since $s' \in A$, Theorem 3.2 is proved. $\square$

Theorem 3.3. *From any ONS one can insolate a subsequence (φ_{n_k}) such that for any function $f \in C_V$,*

$$\sum_{k=1}^{\infty} C_{n_k}^2(f) k^{\gamma} < +\infty,$$

where $C_{n_k}(f) = \int_0^1 f(x)\varphi_{n_k}(x)\,dx$ and $0 < \gamma < 1$.

Proof. Let the ONS (φ_n) be a complete system on $[0,1]$. Then, by the Parceval equality, for any $x \in [0,1]$ we have

$$\sum_{n=1}^{\infty} \left(\int_0^x \varphi_n(u)\,du \right)^2 = x \quad \text{and} \quad \sum_{n=1}^{\infty} \left(\int_0^1 x\varphi_n(u)\,du \right)^2 = \frac{1}{2}.$$

According to the Dini Theorem there exists a sequence of natural numbers (n_k) such that

$$\sum_{s=n_k}^{\infty} \left(\int_0^x \varphi_s(u)\,du \right)^2 < \frac{1}{k^2} \quad \text{and} \quad \sum_{s=n_k}^{\infty} \left(\int_0^1 x\varphi_s(u)\,du \right)^2 < \frac{1}{k^2}$$

uniformly with respect to $x \in [0,1]$. From here, uniformly with respect to $x \in [0,1]$, we obtain

$$\left| \int_0^x \varphi_{n_k}(u)\,du \right| < \frac{1}{k} \quad \text{and} \quad \left| \int_0^1 x\varphi_{n_k}(u)\,du \right| < \frac{1}{k}, \quad k = 1, 2, \ldots. \tag{3.20}$$

In our case let

$$B_m(a,x) = \sum_{k=1}^{m} a_k k^{\gamma} \int_0^x \varphi_{n_k}(t)dt \quad \text{and} \quad H_m(a) = \big(\sum_{k=1}^{m} a_k^2 k^{\gamma}\big)^{\frac{1}{2}}.$$

Next, for arbitrary $(a_n) \in \ell_2$ and $0 < \gamma < 1$ we get (see (3.2) and (3.20))

$$\begin{aligned} D_m(a) &= \max_{1 \le i \le m} \left| \int_0^{\frac{i}{m}} B_m(a,x)\,dx \right| = \left(\int_0^1 B_m^2(a,x)\,dx \right)^{\frac{1}{2}} \\ &\le \left(\sum_{k=1}^{m} a_k^2 k^{\gamma} \right)^{\frac{1}{2}} \left(\sum_{k=1}^{m} k^{\gamma} \left(\int_0^x \varphi_{n_k}(u)\,du \right)^2 \right)^{\frac{1}{2}} \\ &= H_m(a) \left(\sum_{k=1}^{m} k^{\gamma} k^{-2} \right)^{\frac{1}{2}} = O(1) H_m(a). \end{aligned}$$

Thus

$$D_m(a) = O(1) H_m(a). \tag{3.21}$$

In addition (see (3.20)),

$$\begin{aligned} H_m(C(h)) = \left(\sum_{k=1}^{m} C_{n_k}^2(h) k^{\gamma} \right)^{\frac{1}{2}} &= \left(\sum_{k=1}^{m} k^{\gamma} \left(\int_0^1 \varphi_{n_k}(x)\,dx \right)^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{k=1}^{m} k^{\gamma} k^{-2} \right)^{\frac{1}{2}} = O(1) \end{aligned}$$

and

$$H_m(C(g)) = \left(\sum_{k=1}^{m} C_{n_k}^2(g) k^\gamma\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{m} k^\gamma \left(\int_0^1 x\varphi_{n_k}(x)\, dx\right)^2\right)^{\frac{1}{2}}$$

$$= \left(\sum_{k=1}^{m} k^\gamma k^{-2}\right)^{\frac{1}{2}} = O(1).$$

According to (3.21) and Theorem 3.1, for any $f \in C_V$ the series $\sum_{k=1}^{\infty} C_k^2(f) k^\gamma$ is convergent. □

4. Problems of efficiency

Theorem 4.1. *Let (φ_n) be an ONS and*

$$\int_0^x \varphi_n(u)\, du = O(1)\,\frac{1}{n}$$

uniformly with respect to $x \in [0,1]$. Then for arbitrary $(a_n) \in \ell_2$,

$$D_n(a) = O(1) H_n(a). \tag{4.1}$$

Proof. In our case

$$D_n(a) = \max_{1\le i\le n} \left| \int_0^{\frac{i}{n}} B_n(a,x)\, dx \right| = \max_{1\le i\le n} \left| \sum_{k=1}^{n} a_k k^\gamma \int_0^{\frac{i}{n}} \int_0^x \varphi_k(u)\, du\, dx \right|$$

$$= O(1) \sum_{k=1}^{n} \frac{1}{k} |a_k| k^\gamma = O(1) \left(\sum_{k=1}^{n} a_k^2 k^\gamma\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} k^{-2+\gamma}\right)^{\frac{1}{2}} = O(1) H_n(a).$$

So, the trigonometric ($\sqrt{2}\cos 2\pi n x, \sqrt{2}\sin 2\pi n x$) and Walsh systems satisfy condition (4.1). □

Theorem 4.2. *If (X_n) is the Haar system, then for an arbitrary $(a_n) \in \ell_2$,*

$$D_n(a) = O(1) H_n(a).$$

Proof. The definition of the Haar system implies (see [3, Ch. 1])

$$\left| \int_0^x \sum_{k=2^m+1}^{2^{m+1}} a_k k^\gamma X_k(u)\, du \right| \le 2^{-\frac{m}{2}} |a_{k(m)}| k^\gamma(m),$$

where $2^m < k(m) \le 2^{m+1}$.

Without loss of generality, we suppose

$$B_n(a;x) = \sum_{k=2}^{n} a_k k^\gamma \int_0^x \varphi_k(u)\, du.$$

From here, if $n = 2^q$, for an arbitrary $(a_n) \in \ell_2$ $(0 < \gamma < 1)$ we have

$$D_n(a) = \max_{1\le i\le n} \left| \int_0^{\frac{i}{n}} B_n(a,x)\,dx \right|$$
$$= \max_{1\le i\le n} \left| \sum_{m=0}^{q-1} \int_0^{\frac{i}{2^q}} \sum_{k=2^m+1}^{2^{m+1}} \int_0^x X_k(u)\,du\; k^\gamma a_k\,dx \right|$$
$$= O(1) \sum_{m=0}^{q-1} 2^{-\frac{m}{2}} k^\gamma(m) |a_{k(m)}|$$
$$= O(1) \Bigg(\sum_{m=0}^{q-1} \sum_{k=2^m+1}^{2^{m+1}} a_k^2 k^\gamma \Bigg)^{\frac{1}{2}} \Bigg(\sum_{m=0}^{q} 2^{-m} 2^{\gamma m} \Bigg)^{\frac{1}{2}} = O(1) H_n(a).$$

It is easy to prove that when $n = 2^q + l$, $1 \le l \le 2^q$, the condition $D_n(a) = H_n(a)$ is valid. □

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Поступила 23 декабря 2020

После доработки 31 января 2021

Принята к публикации 15 февраля 2021