Известия НАН Армении, Математика, том 57, н. 2, 2022, стр. 14 – 29.

EXPONENTIAL POLYNOMIALS AS SOLUTIONS OF NON-LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

L. K. GAO, K. LIU AND X. L. LIU

https://doi.org/10.54503/0002-3043-2022.57.2-14-29

Nanchang University, Nanchang, P. R. China[1]
University of Eastern Finland, Joensuu, Finland
E-mails: *gaolinkui@126.com; liukai418@126.com, liukai@ncu.edu.cn; liuxinling@ncu.edu.cn*

Abstract. Exponential polynomials, an important subclass of finite order entire functions, as solutions of differential or difference or differential-difference equations are considered in [5, 10, 19, 20]. The critical domains of zeros and the quotients of exponential polynomials are considered in [6]. In this paper, we proceed to consider the exponential polynomials as solutions of some general complex differential-difference equations and extend existence results.

MSC2020 numbers: 30D35; 39A45.
Keywords: exponential polynomial; differential-difference equation; entire solution.

1. Introduction

Assume that the reader is familiar with the standard notation and fundamental results of Nevanlinna theory [4, 8, 22]. A meromorphic function $f(z)$ means meromorphic in the complex plane. If a meromorphic function $f(z)$ has at least one pole, then $f(z)$ is called a properly meromorphic function. Recall the definitions of the order and the hyper-order for a meromorphic function $f(z)$ as follows

$$\sigma(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r},\quad \sigma_2(f)=\limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r}.$$

Exponential polynomials, an important subclass of finite order entire functions with the form

$$f(z)=P_1(z)e^{Q_1(z)}+\cdots+P_k(z)e^{Q_k(z)} \tag{1.1}$$

where $P_j(z)$ and $Q_j(z)$ $(j=1,2,\cdots,k)$ are polynomials in z. It is easy to find that $\sigma(f)=\max\{\deg Q_j\}$ in (1.1). Exponential polynomials are the generalizations of exponential sums which implies that $\max\{\deg Q_j\}=1$ in (1.1). Recently, the exponential polynomial solutions of complex differential or difference or differential-difference equations are considered in [5, 10, 19, 20]. The critical domains of zeros of exponential polynomials and the quotients of exponential polynomials are also

[1]This work was partially supported by the NSFC (No.12061042, 11661052), and the Natural Science Foundation of Jiangxi (No. 20202BAB201003).

considered in [6]. More details on value distribution of exponential sums and exponential polynomials could be seen in [1], [11]–[15].

Let

$$q = \max\{\deg(Q_j) : Q_j(z) \not\equiv 0\},$$

and let $\omega_1, \cdots, \omega_m$ be pairwise different leading coefficients of the polynomials $Q_j(z)$ with the maximum degree q. Thus, (1.1) can be written as

$$f(z) = H_0(z) + H_1(z)e^{\omega_1 z^q} + \cdots + H_m(z)e^{\omega_m z^q}, \tag{1.2}$$

where $H_j(z)$ are either exponential polynomials of degree $< q$ or ordinary polynomials in z. To express the characteristic function of (1.2), we recall the definition of convex hull below.

We fix the notations $W = \{\overline{\omega_1}, \cdots, \overline{\omega_m}\}$, $W_0 = \{0, \overline{\omega_1}, \cdots, \overline{\omega_m}\}$. The convex hull of a set $W \subset \mathbb{C}$, denoted by $co(W)$, is the intersection of all convex sets containing W. If W contains only finitely many elements, then $co(W)$ is obtained as an intersection of finitely closed half-planes, and hence $co(W)$ is either a compact polygon (with a non-empty interior) or a line segment. We denote the perimeter of $co(W)$ by $C(co(W))$. If $co(W)$ is a line segment, then $C(co(W))$ equals to twice the length of this line segment. The following result for exponential polynomials is given by Steinmetz [14].

Theorem A. Let f be given by (1.2). Then

$$T(r, f) = C(co(W_0))\frac{r^q}{2\pi} + o(r^q). \tag{1.3}$$

Yang and Laine [21] investigated the existence of finite order entire solutions $f(z)$ of non-linear differential-difference equations of the form

$$f(z)^n + L(z, f) = h(z),$$

where $L(z, f)$ is a linear differential-difference polynomial, $n \geqslant 2$ is an integer. In particular, Yang and Laine [21] showed that the equation

$$f(z)^2 + q(z)f(z + 1) = P(z), \tag{1.4}$$

has no transcendental entire solutions of finite order, where $P(z)$, $q(z)$ are polynomials. Thus, there does not exist exponential polynomial solutions on (1.4). However, if we replace $q(z)$ with $q(z)e^{Q(z)}$ in (1.4), there exist transcendental entire solutions of finite order. Wen, Heittokangas and Laine [19] studied and classified the finite order entire solutions f of non-linear difference equation

$$f(z)^n + q(z)e^{Q(z)}f(z + c) = P(z). \tag{1.5}$$

For the statement on the properties of transcendental entire solutions below, we give the following notations. Denote

$\Gamma_1 = \{e^{\alpha(z)} + d : d \in \mathbb{C}$ and $\alpha(z)$ is a non-constant polynomial$\}$,

$\Gamma_0 = \{e^{\alpha(z)}$: $\alpha(z)$ is a non-constant polynomial$\}$,

$\Gamma_1' = \{p(z)e^{\alpha(z)} + h(z)$: $p(z) \not\equiv 0$, $h(z)$ are polynomials and $\alpha(z)$ is a non-constant polynomial$\}$,

$\Gamma_0' = \{p(z)e^{\alpha(z)}$: $p(z)$ is a non-zero polynomial and $\alpha(z)$ is a non-constant polynomial$\}$.

Theorem B.[19] *Let $n \geqslant 2$ be an integer, let $c \in \mathbb{C}\backslash\{0\}$ and $q(z)$, $Q(z)$, $P(z)$ be polynomials such that $Q(z)$ is not a constant and $q(z) \not\equiv 0$. Then the finite order transcendental entire solution f of (1.5) satisfies the follows:*

(*a*) *Every solution f satisfies $\sigma(f) = deg(Q)$ and is of mean type.*

(*b*) *Every solution f satisfies $\lambda(f) = \sigma(f)$ if and only if $P(z) \not\equiv 0$.*

(*c*) *A solution f belongs to Γ_0 if and only if $P(z) \equiv 0$. In particular, this is the case if $n \geqslant 3$.*

(*d*) *If a solution f belongs to Γ_0 and if g is any other finite order entire solution to* (1.5)*, then $f = \eta g$, where $\eta^{n-1} = 1$.*

(*e*) *If f is an exponential polynomial solution of the form* (1.1)*, then $f \in \Gamma_1$. Moreover, if $f \in \Gamma_1\backslash\Gamma_0$, then $\sigma(f) = 1$.*

Results in the spirit of Theorem B have been obtained by Li and Yang [9] for more generalized complex difference equation of the form

$$f(z)^n + a_{n-1}f(z)^{n-1} + \cdots + a_1 f(z) + q(z)e^{Q(z)}f(z+c) = P(z), \tag{1.6}$$

where $q(z)$, $P(z)$, $Q(z)$ are polynomials, $n \geqslant 2$ is an integer and $Q(z)$ is not a constant, $q(z) \not\equiv 0$, $c \in \mathbb{C}\backslash\{0\}$ and $a_1, \cdots, a_{n-1} \in \mathbb{C}$.

Note that (1.5) and (1.6) are complex non-linear difference equations. Motivated by (1.5), Liu [10] has classified the finite order entire solutions f of non-linear differential-difference equations of the form

$$f(z)^n + q(z)e^{Q(z)}f^{(k)}(z+c) = P(z), \tag{1.7}$$

where $q(z)$, $P(z)$, $Q(z)$ are polynomials. The results in [19] regarding (1.5) concern the classes Γ_0 and Γ_1. Meanwhile, the results in [10] regarding (1.7) concern the classes Γ_0' and Γ_1'.

We have two motivations as follows and will present some results and discussions in the last two sections.

Motivation 1: Can the results regarding the solutions of (1.5), (1.6) and (1.7) be extended to differential-difference equations, where $f(z+c)$ or $f^{(k)}(z+c)$ is replaced with a differential-difference polynomial?

Motivation 2: How to classify the properly meromorphic solutions of these differential-difference equations?

For the discussions of Motivation 1, it should be very difficult for arbitrary differential-difference polynomials. In this paper, we consider a complex k-homogeneous differential-difference polynomial

$$L(z,f)=\sum_{i=1}^{m}\varphi_i(z)[f^{(\nu_{i1})}(z+c_{i1})]^{k_{i1}}\cdots[f^{(\nu_{in})}(z+c_{in})]^{k_{in}}, \tag{1.8}$$

where $k_{i1}+\cdots+k_{in}=k$, $i=1,\cdots,m$ and $\varphi_i(z)(i=1,\cdots,m)$ are polynomials. We also say $L(z,f)$ has the same shifts, if $c_{i1}=c_{i2}=\cdots=c_{in}$, $i=1,\cdots,m$. For example, $f(z+c)$, $f'(z+c)-f(z+c)$, $f^{(t)}(z+c)$ are 1-homogeneous differential-difference polynomials with the same shifts, $f(z+c)f'(z+c)+f''(z+c)f'''(z+c)$ is a 2-homogeneous differential-difference polynomial with the same shifts.

2. Lemmas

Given a meromorphic function $f(z)$, recall that $\alpha(z)\not\equiv 0,\infty$ is a small function with respect to $f(z)$, if $T(r,\alpha)=S(r,f)$, where $S(r,f)$ is used to denote any quantity satisfying $S(r,f)=o(T(r,f))$, and $r\to\infty$ outside of a possible exceptional set of finite logarithmic measure. The following lemma can be seen as the differential-difference analogue of the logarithmic derivative lemma which is a combination [7, Lemma 2.2] with the lemma on the logarithmic derivative.

Lemma 2.1. *Let f be a transcendental meromorphic function with finite order $\sigma(f)$, let c,h be two complex numbers, $\varepsilon>0$. Then*

$$m\left(r,\frac{f^{(k)}(z+h)}{f(z+c)}\right)=O(r^{\sigma(f)-1+\varepsilon})+O(\log r)=S(r,f). \tag{2.1}$$

Furthermore, if $L(z,f)$ is a k-homogeneous differential-difference polynomial, then

$$m\left(r,\frac{L(z,f)}{f(z)^k}\right)=O(r^{\sigma(f)-1+\varepsilon})+O(\log r)=S(r,f). \tag{2.2}$$

Lemma 2.2. [2, 3] *Let $f(z)$ be a transcendental meromorphic function with $\sigma(f)<\infty$, and let c be a fixed non-zero constant. Then, for each $\varepsilon>0$, we have*

$$T(r,f(z+c))=T(r,f(z))+O(r^{\sigma(f)-1+\varepsilon})+O(\log r).$$

$$N(r,f(z+c))=N(r,f(z))+O(r^{\sigma(f)-1+\varepsilon})+O(\log r).$$

Recall the following two results on the zeros of the 1-homogeneous differential-difference polynomials $f^{(k)}(z)$ and $f(z+c)$.

Lemma 2.3. [22, Theorem 1.24] *Let $f(z)$ be a transcendental meromorphic function and k be a positive integer. Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leqslant T(r, f^{(k)}) - T(r, f) + N(r, \frac{1}{f}) + S(r, f)$$

and

$$N\left(r, \frac{1}{f^{(k)}}\right) \leqslant N(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.4. [9, Lemma 2.3] *Let $f(z)$ be a transcendental meromorphic function with $\sigma_2(f) < 1$, and $c \in \mathbb{C}\backslash\{0\}$. Then*

$$N(r, 1/f(z+c)) = N(r, 1/f) + S(r, f).$$

Related to the zeros of k-homogeneous differential-difference polynomials, we obtain the result below.

Lemma 2.5. *Let $f(z)$ be a transcendental meromorphic function with finite order and $L(z, f)$ be a k-homogeneous differential-difference polynomial. Then*

$$N\left(r, \frac{1}{L(z, f)}\right) \leqslant T(r, L(z, f) - T(r, f) + kN(r, \frac{1}{f}) + S(r, f),$$

and

$$N\left(r, \frac{1}{L(z, f)}\right) \leqslant 2kN(r, \frac{1}{f}) + AN(r, f) + S(r, f),$$

where A is a constant.

Proof. From Lemma 2.1, then

$$m\left(r, \frac{1}{f^k}\right) \leqslant m\left(r, \frac{L(z, f)}{f^k}\right) + m\left(r, \frac{1}{L(z, f)}\right) \leqslant m\left(r, \frac{1}{L(z, f)}\right) + S(r, f). \tag{2.3}$$

Using the first main theorem of Nevanlinna theory, we have

$$T(r, f^k) - N\left(r, \frac{1}{f^k}\right) \leqslant T(r, L(z, f)) - N\left(r, \frac{1}{L(z, f)}\right) + S(r, f).$$

Using Lemma 2.2, we obtain

$$\begin{aligned}
N\left(r,\frac{1}{L(z,f)}\right) &\leqslant T(r,L(z,f))-T(r,f^k)+N\left(r,\frac{1}{f^k}\right)+S(r,f)\\
&= T\left(r,\frac{L(z,f)}{f^k}f^k\right)-T(r,f^k)+N\left(r,\frac{1}{f^k}\right)+S(r,f)\\
&\leqslant T\left(r,\frac{L(z,f)}{f^k}\right)+kN\left(r,\frac{1}{f}\right)+S(r,f)\\
&\leqslant N\left(r,\frac{L(z,f)}{f^k}\right)+kN\left(r,\frac{1}{f}\right)+S(r,f)\\
&\leqslant N(r,L(z,f))+2kN\left(r,\frac{1}{f}\right)+S(r,f)\\
&\leqslant AN(r,f)+2kN\left(r,\frac{1}{f}\right)+S(r,f).
\end{aligned}$$

Remark. (1) If f is a transcendental entire function in Lemma 2.5, then

$$N\left(r,\frac{1}{L(z,f)}\right)\leqslant kN(r,\frac{1}{f})+S(r,f),$$

using the similar reason as the above.

(2) The constant A depends on the expression of $L(z,f)$, which can be obtained by the second equality of Lemma 2.2 and the trivial inequality $N(r,f^{(k)})\leqslant(k+1)N(r,f)+S(r,f)$. For example, $A=3$ for $L(z,f)=f'(z+c)-f(z+c)$.

Lemma 2.6. [4] *Let f be a meromorphic function. Suppose that*

$$\Psi(z):=a_nf(z)^n+\cdots+a_0(z)$$

has small meromorphic coefficients $a_j(z), a_n\neq 0$ in the sense of $T(r,a_j)=S(r,f)$. Moreover, assume that

$$\overline{N}(r,\frac{1}{\Psi})+\overline{N}(r,f)=S(r,f).$$

Then

$$\Psi=a_n\left(f+\frac{a_{n-1}}{na_n}\right)^n.$$

Lemma 2.7. [19] *Let $q\in\mathbb{N}$, $a_0(z),\cdots,a_n(z)$ be either exponential polynomials of degree $<q$ or ordinary polynomials in z, and let $b_1,\cdots,b_n\in\mathbb{C}\backslash\{0\}$ be distinct constants. Then*

$$\sum_{j=1}^{n}a_j(z)e^{b_jz^q}=a_0(z)$$

holds only when $a_0(z)\equiv a_1(z)\equiv\cdots\equiv a_n(z)\equiv 0$.

3. The exponential polynomials solutions on a general equation

Remark that the equations (1.5), (1.6) and (1.7), the last term on the left hand side has only one term $f(z+c)$ or $f^{(k)}(z+c)$. It is natural to ask what will happen $f(z+c)$ or $f^{(k)}(z+c)$ is replaced with differential-difference polynomials. We mainly consider the non-linear differential-difference equations

$$f(z)^n + a_{n-1}f(z)^{n-1} + \cdots + a_s f(z)^s + q(z)e^{Q(z)}[L(z,f)]^t = P(z), \tag{3.1}$$

where $L(z,f)$ is a k-homogeneous differential-difference polynomial. We will assume that $q(z)$, $P(z)$, $Q(z)$ are polynomials, $k \geqslant 1$ is an integer and t is a positive integer, $n > s \geqslant tk \geqslant 1$ and $Q(z)$ is not a constant, $q(z) \not\equiv 0$ and $a_s, \cdots, a_{n-1} \in \mathbb{C}$. It is easy to see that both equations (1.5) and (1.6) have no polynomial solutions, since $Q(z)$ is not a constant. However, there exist polynomial solutions with degree less than k in (1.7) and (3.1), resp. For example, $f(z) = z$ is a solution of

$$f(z)^n - q(z)e^{Q(z)}f''(z+1) = z^n.$$

Recent results on complex differential-difference equations also can be found in [16, 17, 18]. In this paper, we mainly consider the transcendental solutions in (3.1) and obtain the following result.

Theorem 3.1. *The finite order transcendental entire solution f of* (3.1) *should satisfy the following conclusions:*

(*a*) *Every solution f satisfies $\sigma(f) = \deg(Q)$ and is of mean type.*

(*b*) *If $\lambda(f) < \sigma(f)$, then $a_{n-1} = \cdots = a_s = 0$ and $P(z) \equiv 0$.*

(*c*) *If $P(z) \equiv 0$, then $z^{n-s} + a_{n-1}z^{n-s-1} + \cdots + a_s = (z + \frac{a_{n-1}}{n})^{n-s}$. Furthermore, if there exists $i_0 \in \{s, \ldots, n-1\}$ such that $a_{i_0} = 0$, then all the $a_j (j = s, \ldots, n-1)$ must be zeros as well and $\lambda(f) < \sigma(f)$; otherwise $\lambda(f) = \sigma(f)$.*

Furthermore, the following conclusions are true for a k-homogeneous differential-difference polynomial $L(z,f)$ with the same shifts.

(*d*) *$f \in \Gamma_0'$ if and only if $P(z) \equiv 0$ and there exists $i_0 \in \{s, \ldots, n-1\}$ such that $a_{i_0} = 0$.*

(*e*) *If the solution f belongs to Γ_0, then $\sigma(f) = 1$. What's more, if $g \in \Gamma_0$, then $f = \eta g$, where $\eta^{n-kt} = 1$.*

(*f*) *If there exists $i_0 \in \{s, \ldots, n-1\}$ such that $a_{i_0} = 0$. Then*

$$f \in \Gamma_0' \quad \text{and} \quad P(z) \equiv 0 = a_s = \cdots = a_{n-1}$$

provided that one of the following holds:

1) *$s \geqslant k+2$;*

2) *$s = k+1$ and $z^{n-s} + \cdots + a_s = 0$ has at least one zero with multiplicity 2;*

3) $s < k+1$ *and either* $z^{n-s} + \cdots + a_s = 0$ *has at least two zeros with multiplicity 2 or at least one zero with multiplicity 3.*

Remark 3.1. (1) Transcendental entire solutions with finite order of (3.1) exist. For example, the function $f(z) = e^z + \alpha$ solves

$$f(z)^2 - 2\alpha f(z) - 2e^z f'(z - \log 2) = -\alpha^2.$$

(2) From Theorem B (e), we can not get $\sigma(f) = 1$ when f belongs to Γ_0. For example, the function$f(z) = e^{z^2}$ solves

$$f(z)^2 - e^{z^2-2z-1} f(z+1) = 0.$$

However, from Theorem 3.1 (e), if f belongs to Γ_0, the solutions of the equation

$$f(z)^n + a_{n-1} f(z)^{n-1} + \cdots + a_s f(z)^s + q(z)e^{Q(z)}[L(z,f)]^t = 0,$$

must satisfy $\sigma(f) = 1$. Such solutions exist, for example, the function $f(z) = e^z$ solves $f(z)^2 - e^{z-1} f'(z+1) = 0$.

Proof of Theorem 3.1 (a). Assume that $f(z)$ is a finite order transcendental entire solution of (3.1). From Valiron-Mohon'ko theorem and Lemma 2.1, we obtain

$$\begin{aligned} nT(r,f) &= T(r, f^n + \cdots + a_s f^s) + S(r,f) = T(r, P(z) - q(z)e^{Q(z)}[L(z,f)]^t) + S(r,f) \\ &= m(r, P(z) - q(z)e^{Q(z)}[L(z,f)]^t) + S(r,f) \\ &\leqslant m(r, P(z)) + m(r, q(z)) + m(r, e^{Q(z)}) + m(r, [L(z,f)]^t) + S(r,f) \\ &\leqslant m(r, e^{Q(z)}) + m\left(r, \left(\frac{L(z,f)}{f(z)^k}\right)^t f(z)^{tk}\right) + S(r,f) \\ &\leqslant m(r, e^{Q(z)}) + tkT(r, f(z)) + S(r,f). \end{aligned}$$

Since $n > tk$, then

$$(n - tk)T(r,f) \leqslant m(r, e^{Q(z)}) + S(r,f),$$

which implies that $\sigma(f) \leqslant \deg(Q(z))$. If $\sigma(f) < \deg(Q(z))$, then $\sigma(L(z,f)) < \deg(Q(z))$, which is impossible for (3.1). Hence, $\sigma(f) = \deg(Q(z))$. From the definition of type, we get

$$\tau(f) = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\deg(Q(z))}} \in (0, +\infty),$$

which implies that $f(z)$ is of mean type.

Proof of Theorem 3.1 (b). If $\lambda(f) < \sigma(f)$, that is the value 0 is a Borel exceptional value of $f(z)$, then f is of regular growth (or normal growth) [22, Theorem 2.11]. Thus, $N(r, \frac{1}{f}) = S(r,f)$ follows by [22, Theorem 1.18]. Let

(3.2) $G(z) := f(z)^n + a_{n-1} f(z)^{n-1} + \cdots + a_s f(z)^s - P(z) = -q(z)e^{Q(z)}[L(z,f)]^t.$

From the Remark after Lemma 2.5 and f is an entire function, we have

$$
\begin{aligned}
N\left(r,\frac{1}{G(z)}\right) &= N\left(r,\frac{1}{q(z)[L(z,f)]^t}\right)\\
&\leqslant N\left(r,\frac{1}{q(z)}\right)+tN\left(r,\frac{1}{L(z,f)}\right)\leqslant tkN(r,\frac{1}{f})+S(r,f)=S(r,f).
\end{aligned}\tag{3.3}
$$

Thus $\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,f)=S(r,f)$. Lemma 2.6 implies that

$$G(z)=\left(f+\frac{a_{n-1}}{n}\right)^n. \tag{3.4}$$

If $a_{n-1}\neq 0$, using the second main theorem of Nevanlinna theory, we have

$$
\begin{aligned}
T(r,f) &\leqslant \overline{N}\left(r,\frac{1}{f+\frac{a_{n-1}}{n}}\right)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f)\\
&\leqslant \overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f)=S(r,f),
\end{aligned}
$$

a contradiction. Thus $a_{n-1}=0$. Therefore, from (3.4) and (3.2), we have

$$a_{n-1}=\cdots=a_s=0\equiv P(z).$$

Proof of Theorem 3.1 (c). Since $P(z)\equiv 0$ and $s\geqslant tk$, then (3.1) can be written as

$$
\begin{aligned}
H(z) &:= f(z)^{n-kt}+a_{n-1}f(z)^{n-kt-1}+\cdots+a_sf(z)^{s-kt}\\
&= -q(z)e^{Q(z)}\left[\frac{L(z,f)}{f(z)^k}\right]^t.
\end{aligned}\tag{3.5}
$$

From (3.5), we have

$$tN\left(r,\frac{L(z,f)}{f(z)^k}\right)\leqslant N\left(r,\frac{1}{q(z)}\right)=S(r,f). \tag{3.6}$$

Combining (3.6) with Lemma 2.1, we obtain

$$T\left(r,\frac{L(z,f)}{f(z)^k}\right)=S(r,f). \tag{3.7}$$

Using the first main theorem of Nevanlinna theory, we have

$$T\left(r,\frac{1}{\frac{L(z,f)}{f(z)^k}}\right)=T\left(r,\frac{L(z,f)}{f(z)^k}\right)+O(1)=S(r,f). \tag{3.8}$$

From (3.5) and (3.8), we obtain

$$\overline{N}\left(r,\frac{1}{H(z)}\right)+\overline{N}(r,f)\leqslant N\left(r,\frac{1}{\frac{L(z,f)}{f(z)^k}}\right)+N\left(r,\frac{1}{q(z)}\right)=S(r,f).$$

Lemma 2.6 implies that

$$H(z)=\left(f(z)+\frac{a_{n-1}}{n-kt}\right)^{n-kt}=-q(z)e^{Q(z)}\left(\frac{L(z,f)}{f(z)^k}\right)^t. \tag{3.9}$$

Case 1. There exists $i_0 \in \{s, \cdots, n-1\}$ such that $a_{i_0} = 0$. From (3.9), we have $a_{n-1} = \cdots = a_s = 0$. Thus, (3.1) can be reduced to the follows form

$$f(z)^{n-kt} = -q(z)e^{Q(z)}\left(\frac{L(z,f)}{f(z)^k}\right)^t. \tag{3.10}$$

Since $n > kt$, then $N(r, \frac{1}{f}) = S(r, f)$. So $\lambda(f) < \sigma(f)$.

Case 2. There does not exist $i_0 \in \{s, \cdots, n-1\}$ such that $a_{i_0} = 0$. From (3.8) and (3.9), we have

$$\overline{N}\left(r, \frac{1}{f(z) + \frac{a_{n-1}}{n-kt}}\right) \leqslant \overline{N}\left(r, \frac{1}{\frac{L(z,f)}{f(z)^k}}\right) + \overline{N}\left(r, \frac{1}{q(z)}\right) = S(r, f).$$

Using the second main theorem, we have

$$\begin{aligned} T(r,f) &\leqslant \overline{N}\left(r, \frac{1}{f(z) + \frac{a_{n-1}}{n-kt}}\right) + \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}(r, f(z)) + S(r, f) \\ &= \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r, f). \end{aligned}$$

Therefore, $\lambda(f) = \sigma(f)$.

Proof of Theorem 3.1 (d). If $f \in \Gamma_0'$, then $\lambda(f) < \sigma(f)$ follows. From Theorem 3.1 (b), we have $a_{n-1} = \cdots = a_s = 0 \equiv P(z)$.

On the other hand, we will prove that if $P(z) \equiv 0$ and there exists $i_0 \in \{s, \ldots, n-1\}$ such that $a_{i_0} = 0$, then $f \in \Gamma_0'$. The condition $P(z) \equiv 0$ implies that (3.1) reduces to

$$f(z)^n + q(z)e^{Q(z)}[L(z, f)]^t = 0. \tag{3.11}$$

Since $P(z) \equiv 0$ and there exists an $i_0 \in \{s, \ldots, n-1\}$ such that $a_{i_0} = 0$, from Theorem 3.1 (c), then $\lambda(f) < \sigma(f)$. The Hadamard factorization theorem implies that

$$f(z) = H(z)e^{\alpha(z)}, \tag{3.12}$$

where $\alpha(z)$ is a non-constant polynomial with $\deg(\alpha(z)) = \deg(Q(z))$ and $H(z)$ is an entire function satisfying

$$\lambda(H(z)) = \sigma(H(z)) = \lambda(f) < \sigma(f).$$

In the following, we will prove $H(z)$ is a non-zero polynomial. Otherwise, if $H(z)$ is a transcendental entire function, from (3.12), then

$$f^{(v_{ij})}(z + c) = H_{ij}(z + c)e^{\alpha(z+c)} \quad (j \in 1, 2, \cdots, n),$$

where $H_{ij}(z + c)$ is 1-homogeneous differential-difference polynomial in $H(z + c)$. Remark that $L(z, f)$ is a k-homogeneous differential-difference polynomial with the same shifts in (1.8), then

$$\begin{aligned} L(z,f) &= \sum_{i=1}^{m} H_{i1}(z+c)\cdots H_{in}(z+c)e^{k\alpha(z+c)} \\ &= e^{k\alpha(z+c)}\sum_{i=1}^{m} H_{i1}(z+c)\cdots H_{in}(z+c) =: e^{k\alpha(z+c)}H_k(z) \end{aligned}$$

where $\sigma(H_k(z)) < \sigma(f)$ and $H_k(z)$ is a k-homogeneous differential-difference polynomial.

Substituting (3.12) and (3.13) into (3.11), we have

$$H(z)^n e^{n\alpha(z)} + q(z)e^{Q(z)+kt\alpha(z+c)}H_k(z)^t = 0, \tag{3.13}$$

so

$$H(z)^{n-kt} + q(z)e^{Q(z)+kt\alpha(z+c)-n\alpha(z)}\left(\frac{H_k(z)}{H(z)^k}\right)^t = 0. \tag{3.14}$$

By Lemma 2.1, we have

$$m\left(r, \frac{H_k(z)}{H(z)^k}\right) = O(r^{\sigma(H)-1+\varepsilon}). \tag{3.15}$$

From (3.14), the poles of $\frac{H_k(z)}{H(z)^k}$ are the zeros of $q(z)$, then

$$N\left(r, \frac{H_k(z)}{H(z)^k}\right) = O(\log r). \tag{3.16}$$

Thus,

$$T\left(r, \frac{H_k(z)}{H(z)^k}\right) = O(r^{\sigma(H)-1+\varepsilon}) + O(\log r). \tag{3.17}$$

Hence, we have $N\left(\frac{1}{\frac{H_k(z)}{H(z)^k}}\right) = O(r^{\sigma(H)-1+\varepsilon}) + O(\log r)$. Since $n > kt$, so the zeros of $H(z)$ are the zeros of $\frac{H_k(z)}{H(z)^k}$ or $q(z)$, thus $\lambda(H) < \sigma(H)$, which is a contradiction with $\lambda(H) = \sigma(H)$. So $H(z)$ is a polynomial. Hence, $f \in \Gamma_0'$.

Proof of Theorem 3.1 (e). If $f, g \in \Gamma_0$, then $a_{n-1} = \cdots = a_s = 0 \equiv P(z)$ follows by Theorem 3.1 (b) and $H(z) \equiv 1$ in (3.12). From (3.13), we have $q(z)$, $H_k(z)$ are also constants. If $H_k(z)$ is a constant, then $\alpha(z)$ must be a linear polynomial and $\varphi_i(z)$ are also constants φ_i. We may assume that $q(z) = q \in \mathbb{C}$ and $f(z) = e^{b_1 z + d_1}$ and $g(z) = e^{b_2 z + d_2}$, where $b_i (\neq 0), d_i$ $(i = 1, 2)$ are constants. Substituting $f(z)$ and $g(z)$ into (3.11), we can get

$$e^{nb_1 z + nd_1} + q(z)e^{Q(z)}L_1(b_1)e^{ktb_1 z + kt(d_1+c)} = 0, \tag{3.18}$$

and

$$e^{nb_2 z + nd_2} + q(z)e^{Q(z)}L_1(b_2)e^{ktb_2 z + kt(d_2+c)} = 0, \tag{3.19}$$

where $L_1(z) = \sum_{i=1}^{m} \varphi_i z^{k_{i1}v_{i1}} \cdots z^{k_{in}v_{in}}$. Combining (3.18) and (3.19), we have

$$L_1(b_1)e^{(kt-n)b_1 z + (kt-n)d_1} \equiv L_1(b_2)e^{(kt-n)b_2 z + (kt-n)d_2}.$$

Thus, we have $b_1 = b_2$ and $e^{(n-kt)(d_1-d_2)} = 1$, which implies that $f = \eta g$, where $\eta^{n-kt} = 1$.

Proof of Theorem 3.1 (f). If $P(z) \not\equiv 0$, from Lemma 2.1, Lemma 2.2 and the second main theorem of Nevanlinna theory, then

(3.20)

$$\begin{aligned}
&nT(r,f) = T\left(r, f^n + \cdots + a_s f^s\right) + S(r,f)\\
&\leqslant \overline{N}\left(r, \frac{1}{f^n + \cdots + a_s f^s - P(z)}\right) + \overline{N}\left(r, \frac{1}{f^n + \cdots + a_s f^s}\right) + \overline{N}\left(r, f^n + \cdots + a_s f^s\right) + S(r,f)\\
&\leqslant \overline{N}\left(r, \frac{1}{f^n + \cdots + a_s f^s}\right) + \overline{N}\left(r, \frac{1}{q(z)[L(z,f)]^t}\right) + S(r,f)\\
&\leqslant kT(r,f) + \overline{N}\left(r, \frac{1}{f^n + \cdots + a_s f^s}\right) + S(r,f).
\end{aligned}$$

We will get a contradiction from (3.20) in every case of 1), 2), 3) to show $P(z) \equiv 0$ below. Thus, the conclusion of (f) follows by Theorem 3.1 (c) and (d).

Case 1). If $s \geqslant k+2$, from (3.20), we have

$$\begin{aligned}
(n-k)T(r,f) &\leqslant \overline{N}\left(r, \frac{1}{f(z)^s(f(z)^{n-s} + \cdots + a_s)}\right) + S(r,f)\\
&\leqslant (n-s+1)T(r,f) + S(r,f),
\end{aligned}$$

which is a contradiction.

Case 2). If $s = k+1$ and $z^{n-s} + \cdots + a_s = 0$ has at least one zero with multiplicity 2. From (3.20), we have

$$(n-k)T(r,f) \leqslant (n-s)T(r,f) + S(r,f),$$

a contradiction.

Case 3). If $s \leqslant k+1$ and $z^{n-s} + \cdots + a_s = 0$ has at least two zeros with multiplicity 2 or at least one zero with multiplicity 3, we also get a contradiction from (3.20).

4. Properly meromorphic solutions

In this section, we will consider the properly meromorphic solutions on non-linear differential-difference equations.

Wen etc. [19] proved that there is no properly meromorphic solutions with hyper-order less than one on (1.5) by considering the poles multiplicities provided that $n \geqslant 2$, which is also true for (1.6). In fact, we know that both (1.5) and (1.6) has no any properly meromorphic solutions by the follows statements. We assume that $f(z)$ is a properly meromorphic solution of (1.5) or (1.6) and $f(z)$ has a pole z_0, then $z_k = z_0 + kc$ are also the poles of $f(z)$, where k is any integer, thus $f(z)$ should have infinitely many poles. Let $\mathbb{E}$ be the set of all poles multiplicities of $f(z)$ and m

be the minimum of $\mathbb{E}$, where m is called the index of $f(z)$. Obviously, $f(z+c)$ has the same index m, but the index of $f(z)^n$ is mn, which is impossible for (1.5) or (1.6), when $n \geqslant 2$. Thus, $f(z)$ has no poles. Similarly, if $L(z,f)$ is a linear difference polynomial, then (3.1) has no properly meromorphic solutions.

Remark that if $L(z,f)$ includes the derivatives of $f(z)$ or the derivatives of $f(z+c)$, then (3.1) may have properly meromorphic solutions, which can be seen by the examples below.

Examples (1) Properly meromorphic function $f(z) = \frac{1}{1-e^z}$ solves the follows two equations

$$f(z)^2 - e^{-z} f'(z+2\pi i) = 0, \qquad f(z)^2 - e^{-z} f'(z) = 0.$$

(2) Properly meromorphic function $f(z) = \frac{1}{e^z+1}$ solves

$$f(z)^3 - \frac{5}{2} f(z)^2 + 2f(z) + \frac{1}{2} e^z f''(z) = \frac{1}{2}.$$

and $f(z) = \frac{1-e^z}{2(e^z+1)}$ solves

$$f(z)^3 - f(z)^2 + \frac{1}{4} f(z) + \frac{1}{2} e^z f''(z) = 0.$$

(3) Properly meromorphic solutions with infinite order of (3.1) also exist. For example, the function $f(z) = \frac{1}{e^{-e^{-z}}-1}$ solves

$$f(z)^2 + f(z) + e^z f'(z) = 0.$$

and the function $f(z) = \frac{3e^{-e^{-z}}-2}{e^{-e^{-z}}-1}$ solves

$$f(z)^2 - 5f(z) - e^z f'(z) = -6.$$

Remark that some functions are periodic functions in the above examples, so $f'(z)$ can be replaced with $f'(z+c)$ in the above equations for suitable constants c. An elementary calculation to find that the non-linear differential equation

$$f(z)^2 + e^{Q(z)} f^{'}(z) = 0 \tag{4.1}$$

has solutions $f(z) = \frac{1}{C+\int e^{-Q(z)}dz}$, C is a constant. In addition, the solutions of

$$f(z)^2 + a_1 f(z) + q(z) e^{Q(z)} f'(z) = 0 \tag{4.2}$$

can be expressed by $f = \frac{1}{e^{\int \frac{a_1}{q(z)} e^{-Q(z)} dz} \left(\int \frac{e^{-Q(z)}}{q(z)} e^{\int \frac{-a_1}{q(z)} e^{-Q(z)} dz} dz + C \right)}$.

Question 1: How to classify the properly meromorphic solutions of

$$f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + q(z) e^{Q(z)} L(z,f) = P(z), \tag{4.3}$$

where $L(z,f)$ is a k-homogeneous differential-difference polynomial, $k \geqslant 1$ is an integer and $q(z)$, $P(z)$, $Q(z)$ are polynomials, $a_1, \cdots, a_{n-1}$ are constants.

In the paper, using the exponential polynomials, we consider the simple case of $n=2$, $L(z,f)=f'(z)$ and $P(z)\equiv 0$ in (4.3), that is

$$f^2+a_1f+q(z)e^{Q(z)}f'(z)=0, \tag{4.4}$$

where $Q(z)=b_qz^q+\cdots+b_0,\ b_q\neq 0,\ a_1$ is a constant and $q(z)$ is a non-zero polynomial.

It is easy to find that the meromorphic solutions of (4.4) have only finitely many zeros. For the simplified expressions, we can consider the meromorphic solutions f with the form

$$f(z)=\frac{1}{g(z)}=\frac{1}{G_0(z)+G_1(z)e^{\omega_1 z^q}+\cdots+G_m(z)e^{\omega_m z^q}},$$

where $g(z)$ is an exponential polynomial and $G_j(z)(j=0,1,\cdots,m)$ are either exponential polynomials of degrees $<q$ or ordinary polynomials in z.

Theorem 4.1. *(i) If $a_1=0$, then (4.4) admits properly meromorphic solutions of the form $f=\frac{1}{g}=\frac{1}{d+Ae^{\omega_1 z}}$, where d,A,ω_1 are constants.*

(ii) If $a_1\neq 0$, then (4.4) has no meromorphic solutions of the form $f=\frac{1}{g}$.

Proof of Theorem 4.1. (i) If $a_1=0$, substitute $f=\frac{1}{g}$ and

$$g(z)=G_0(z)+G_1(z)e^{\omega_1 z^q}+\cdots+G_m(z)e^{\omega_m z^q}$$

into (4.4), then

$$1-q(z)e^{Q_0(z)}(G_0'(z)e^{b_qz^q}+G_{1,1}(z)e^{(\omega_1+b_q)z^q}+\cdots+G_{m,1}(z)e^{(\omega_m+b_q)z^q})=0, \tag{4.5}$$

where $Q_0(z)=Q(z)-b_qz^q$ is a polynomial of degree $\leqslant q-1$ and

$$G_{k,1}(z)=G_k'(z)+q\omega_kz^{q-1}G_k(z)\not\equiv 0 \tag{4.6}$$

for $k=1,\cdots,m$.

If $m\geqslant 2$, from Lemma 2.7 and (4.5), we get that at least one of $q(z)e^{Q_0(z)}G_{1,1}(z)$ and $q(z)e^{Q_0(z)}G_{m,1}(z)$ is equal to zero, thus $G_{1,1}(z)$ or $G_{m,1}(z)$ is equal to zero, which is impossible.

If $m=1$, then (4.5) reduces to

$$1-q(z)e^{Q_0(z)}\left(G_0'(z)e^{b_qz^q}+G_{1,1}(z)e^{(\omega_1+b_q)z^q}\right)=0. \tag{4.7}$$

Let $h_1(z)=q(z)e^{Q_0(z)}G_0'(z)e^{b_qz^q}$ and $h_2(z)=q(z)e^{Q_0(z)}G_{1,1}(z)e^{(\omega_1+b_q)z^q}$. Thus, $h_1(z)+h_2(z)=1$. Using the second main theorem of Nevanlinna theory for $h_1(z)$, we have $h_1(z)$ and $h_2(z)$ must be constants. Hence, $G_0'(z)=0$ and $\omega_1=-b_q$ by the expressions of $h_1(z)$ and $h_2(z)$. Let $g(z)=d+G_1(z)e^{\omega_1 z^q}$. We proceed to prove that $q=1$. Otherwise, (4.7) can be written as

$$q(z)e^{Q(z)+\omega_1 z^q}(G_1'(z)+G_1(z)q\omega_1z^{q-1})=1. \tag{4.8}$$

(4.8) means that $q(z)$ is a non-zero constant. Furthermore, $Q(z)+\omega_1 z^q$ and $G_1'(z)+G_1(z)q\omega_1 z^{q-1}$ must be constants, otherwise $G_1(z)$ is of order q. Hence, $q=1$ and $G_1(z)$ is a non-zero constant A. Thus, $g(z)=d+Ae^{\omega_1 z}$.

(ii) If $a_1 \neq 0$, substitute $f=\frac{1}{g}$ into (4.4), we have

$$1+a_1(G_0(z)+G_1(z)e^{\omega_1 z^q}+\cdots+G_m(z)e^{\omega_m z^q})-q(z)e^{Q_0(z)}(G_0'(z)e^{b_q z^q}+$$

$$G_{1,1}(z)e^{(\omega_1+b_q)z^q}+\cdots+G_{m,1}(z)e^{(\omega_m+b_q)z^q})=0, \tag{4.9}$$

where $G_{k,1}(z)$ are the same as (4.6). If $G_0'(z)\neq 0$, then b_q, ω_1+b_q, $\cdots$, ω_m+b_q are $m+1$ distinct constants, hence $\{\omega_1,\cdots,\omega_m\}\neq\{b_q,\omega_1+b_q,\cdots,\omega_m+b_q\}$, then there exists $i\in\{0,1,2,\cdots,m\}$ such that $\omega_i+b_q\neq\omega_j$, $j=1,\cdots,m$ and $\omega_0=0$. By Lemma 2.7, we have $G_{i,1}(z)\equiv 0$, which is a contradiction. Thus, we have $G_0'(z)=0$ and $\{\omega_1,\cdots,\omega_m\}$=$\{\omega_1+b_q,\cdots,\omega_m+b_q\}$ for $m\geqslant 1$, which is also impossible unless $b_q=0$.

Remark 4.1. (1) The case (i) shows that all properly meromorphic solutions with the form $f(z)=\frac{1}{g}$ are of order 1. However, if $a_1=0$ and $q(z)$ is a rational function, then (4.4) has properly meromorphic solutions f with finite order $\sigma(f)>1$. For example, the function $f(z)=\frac{1}{e^{z^n}+1}$ solves

$$f(z)^2+\frac{1}{nz^{n-1}}e^{-z^n}f'(z)=0.$$

(2) If $P(z)\not\equiv 0$ in (4.3), the equation

$$f^2+a_1 f+q(z)e^{Q(z)}f'(z)=P(z) \tag{4.10}$$

may admit properly meromorphic solutions of the form $f=\frac{h}{g}$, where $h(z)$ and $g(z)$ are exponential polynomials. The functions $f_1(z)=\frac{1}{e^z-1}$ and $f_2(z)=\frac{1-e^z}{2e^z-1}$ solve

$$f(z)^2+2f(z)+e^z f'(z)=-1.$$

Hence, to classify the general ratios of exponential polynomials for non-linear differential-difference equation is deserved to considering.

(3) We have the basic discussions on $L(z,f)=f'(z)$ in (4.4). However, if $L(z,f)$ includes differential-difference polynomials, for example $L(z,f)=f'(z+c)$, the corresponding substitution will be more complicated than (4.5). In this case, it is not clear for the expressions of properly meromorphic solutions of (4.3).

Acknowledgments. The authors would like to thank the referee for his/her helpful comments and suggestions.

Список литературы

[1] C. Berenstein and R. Gay, Complex Analysis and Special Topic in Harmonic Analysis, Springer (1995).

[2] Z. X. Chen, Complex Differences and Difference Equations, Mathematics Monograph Series, **29**, Science Press, Beijing (2014).
[3] Y. M. Chiang and S. J. Feng, "On the Nevanlinna characteristic $f(z + \eta)$ and difference equations in the complex plane", The Ramanujan. J. **16**, 105 – 129 (2008).
[4] W. K. Hayman, Meromorphic Functions, Clarendon Press, Oxford (1964).
[5] J. Heittokangas, I. Laine, K. Tohge and Z. T. Wen, "Completely regular growth solutions of second order complex linear differential equations", Ann. Acad. Sci. Fenn. Math., **40**, 985 – 1003 (2015).
[6] J. Heittokangas, K. Ishizaki, K. Tohge and Z. T. Wen, "Zero distribution and divison results for exponential polynomials", Isr. J. Math. **227**, 397 – 421 (2018).
[7] R. Korhonen, "An extension of Picard's theorem for meromorphic functions of small hyper-order", J. Math. Anal. Appl., **357**, 244 – 253 (2009).
[8] I. Laine, Nevanlinna Theory and Complex Differential Equations, Walter de Gruyter, Berlin-New York (1993).
[9] N. Li and L. Z. Yang, "Solutions of nonlinear difference equations", J. Math. Anal. Appl., **452**, 1128 – 1144 (2017).
[10] K. Liu, "Exponential polynomials as solutions of differential-difference equations of certain type", Mediterr. J. Math. **13**, 3015 – 3027 (2016).
[11] G. Pólya, "Geometrisches über die Verteilung der Nullstellen spezieller ganzer Funktionen.", Sitz.-Ber. Bayer. Akad. Wiss., 285 – 290 (1920).
[12] G. Pólya, "Untersuchungen über Lücken und Singularitäten von Potenzreihen", Math. Z. **29**, no. 1, 549 – 640 (1929).
[13] E. Schwengeler, Geometrisches über die Verteilung der Nullstellen spezieller ganzer Funktionen (Exponentialsummen), Diss. Zürich (1925).
[14] N. Steinmetz, "Wertverteilund von exponentialpolynomen", Manuscripta Math. **26**, 155 – 167 (1978).
[15] N. Steinmetz, "Zur Wertverteilund der quotienten von exponentialpolynomen", Arch. Math. **35**, 461 – 470 (1980).
[16] Q. Wang and Q. Y. Wang, "Study on the existence of solutions to two special types of differential-difference equations", Turk. J. Math., **43**, no.5, 941 – 954 (2019).
[17] Q. Y. Wang, "Admissible meromorphic solutions of algebraic differential-difference equations", Math. Meth. Appl. Sci. 1 – 10 (2019). https://doi.org/10.1002/mma.5564.
[18] Q. Y. Wang, G. P. Zhan and P. C. Hu, "Growth on meromorphic solutions of differential-difference equations", Bull. Malays. Math. Sci. Soc, **43**, 1503 – 1515 (2020).
[19] Z. T. Wen, J. Heittokangas and I. Laine, "Exponential polynomials as solutions of certain nonlinear difference equations", Acta Math. Sin. **28**, 1295 – 1306 (2012).
[20] Z. T. Wen, G. G. Gundersen and J. Heittokangas, "Dual exponential polynomials and linear differential equations", J. Diff. Equa., **264**, 98 – 114 (2018).
[21] C. C. Yang and I. Laine, "On analogies between nonlinear difference and differential equations", Proc. Japan. Acad. Ser. A Math. Sci., **86**(1), 10 – 14 (2010).
[22] C. C. Yang and H. X. Yi, Uniqueness Theory of Meromorphic Functions, Kluwer Academic Publishers (2003).

Поступила 19 марта 2021

После доработки 14 мая 2021

Принята к публикации 31 октября 2021