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ON THE SUMMABILITY OF FOURIER SERIES BY THE GENERALIZED CESÁRO METHOD

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Abstract. The analogous of Lebesgue-Gergen convergence test for generalized Cesáro means of Fourier trigonometric series is given.

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1. Notations and formulation of the Main Theorem

Let f be a 2π-periodic locally integrable function and

$$S_n(x,f) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

be the partial sums of the Fourier series of f with respect to the trigonometric system.

Let (α_n) be a sequence of real numbers, where $\alpha_n > -1$, $n = 1, 2, \dots$. Suppose

$$\sigma_n^{\alpha_n}(x,f) =: \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha_n - 1} S_\nu(x,f)/A_n^{\alpha_n},$$

where

$$A_k^{\alpha_n} = (\alpha_n + 1)(\alpha_n + 2) \dots (\alpha_n + k)/k!.$$

These means (generalized Cesáro (C, α_n) means) were introduced by Kaplan [7]. The author compared the methods of summability (C, α_n) and (C, α) for number series, and obtained necessary and sufficient conditions, in terms of the α_n, for the inclusion $(C, \alpha_n) \subset (C, \alpha)$, and sufficient conditions for $(C, \alpha) \subset (C, \alpha_n)$. Later Akhobadze ([1]-[5]) and Tetunashvili [10]-[15] investigated problems of (C, α_n) summability of trigonometric Fourier series.

If (α_n) is a constant sequence ($\alpha_n = \alpha$, $n = 1, 2, \dots$) then $\sigma_n^{\alpha_n}(x,f)$ coincides with the usual Cesáro $\sigma_n^{\alpha}(x,f)-$means [18, Ch. III].

One of the most general test of convergence of Fourier series at a point was given by Lebesgue [8].

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Theorem 1.1 (Lebesgue). *Let f be 2π-periodic locally integrable function ($f \in L([0,2\pi])$) and at a point x the following conditions are fulfilled:*

$$h^{-1}\int_0^h |\varphi(x,t)|\,dt = o(1) \tag{1.1}$$

and

$$\int_h^\pi t^{-1}\,|\varphi(x,t)-\varphi(x,t+h)|\,dt = o(1),\ \ h\to +0, \tag{1.2}$$

where

$$\varphi(x,t) = f(x+t)+f(x-t)-2f(x). \tag{1.3}$$

Then the trigonometric Fourier series convergence at the point x.

In 1930 Gergen [6] improved the last Lebesgue statement. In particular, he proved

Theorem 1.2 (Gergen). *Let*

$$\Phi(x,t) = \int_0^t \varphi(x,u)du.$$

If $f \in L([0,2\pi])$ and at a point x relations (1.2) *and*

$$h^{-1}\Phi(x,h) = o(1),\ h\to +0, \tag{1.4}$$

are valid, then the Fourier series of f convergence at the point x.

In 1981 Sahney and Waterman [9] proved

Theorem 1.3 (Sahney, Waterman). *Let $-1<\alpha<0$. Suppose that assumption* (1.1) *holds true and*

$$\int_\eta^\pi t^{-1-\alpha}\,|\varphi(x,t)-\varphi(x,t+\eta)|\,dt = o(n^\alpha),\ \ \eta = \pi/(n+(\alpha+1)/2)\to +0.$$

Moreover, let

$$\Phi(x,\pi)-\Phi(x,\pi-h) = o(h^{-\alpha}),\ \ h\to +0. \tag{1.5}$$

Then the trigonometric Fourier series is (C,α)-summable at x.

Long ago (in 1964) Zhizhiashvili ([16]; see, also, [17, Theorem 2.2.1]), proved more strong result then the last theorem. In particular, he showed that condition (1.5) is not necessary.

Theorem 1.4 (Zhizhiashvili)**.** *Suppose* $-1<\alpha<1$. *Then under assumptions* (1.4) *and*

$$h^{\alpha}\int_{h}^{\pi} t^{-1-\alpha}\left|\varphi(x,t)-\varphi(x,t+h)\right|dt=o(1),\ h\to+0, \tag{1.6}$$

the Fourier series of f *is* (C,α)*-summable at point* x.

The object of this paper is to generalize the above result for (C,α_n)-summability method.

Theorem 1.5. *Let* $-1<\alpha_n<1,\ n=1,2,...,$ *and*

$$\overline{\Phi}(x,t)=\sup_{0\le u\le t}\left|\Phi(x,u)\right|.$$

Suppose that

$$\frac{1}{(1+\alpha_n)n}\int_{\frac{\pi}{n}}^{\pi}\frac{\overline{\Phi}(x,t)}{t^3}dt=o(1), \tag{1.7}$$

$$\frac{1}{(1+\alpha_n)n^{\alpha_n}}\sup_{0<h\le\frac{\pi}{n}}\int_{\frac{\pi}{n}}^{\pi} t^{-1-\alpha_n}\left|\varphi(x,t)-\varphi(x,t+h)\right|dt=o(1),\ n\to\infty, \tag{1.8}$$

hold true. Then the trigonometric Fourier series is (C,α_n) *-summable at* x. *Summability is uniform over any closed interval inside interval of continuity where* (1.7) *and* (1.8) *are satisfied uniformly.*

Using the last statement it is easy to prove

Corollary 1.1. *Let* $\alpha_0\in[0,1)$ *and for all* n *natural number* $\alpha_n\in(\alpha_0,1)$. *Then for almost all* x *the trigonometric Fourier series is* $(C,-\alpha_n)$*-summable at point* x.

Corollary 1.2. *Theorem* 1.4 *in the case* $-1<\alpha\le 0$ *is a consequence of Theorem* 1.5.

2. Auxiliary statements

Let $K_n^{\alpha_n}(t)$ be the kernel of the (C,α_n)-summability method.

Lemma 2.1. [3, Lemma 2] *For every natural* n *and* $\alpha_n\in(-1,1)$

$$\left|K_n^{\alpha_n}(x)\right|\le\frac{n}{1+\alpha_n}+\frac{1}{2}. \tag{2.1}$$

Lemma 2.2. *If* k,n *and* i *are natural numbers then*

$$C_1(i)(i+\alpha_n)(i+1+\alpha_n)k^{\alpha_n}<A_k^{\alpha_n}<C_2(i)(i+\alpha_n)(i+1+\alpha_n)k^{\alpha_n},$$

$\alpha_n \in (-i-1, -i)$.[2]

This lemma actually was proved in [3, Lemma 2].

Lemma 2.3. *For every natural n and $\alpha_n \in (-1, 1)$*

$$\left|(K_n^{\alpha_n}(x))'\right| \leq \frac{4n^2}{1+\alpha_n}.$$

Proof. The proof of this lemma is a simple consequence of Jackson's well-known inequality (see, e.g., [18, Ch. III, Lemma (13.16)]) and Lemma 2.1. □

Using representation (1.12) (see [19, Ch. XI]) for sequence (α_n) we get

$$K_n^{\alpha_n}(t) = \varphi_n^{\alpha_n}(t) + r_n^{\alpha_n}(t),$$

where

$$\varphi_n^{\alpha_n}(t) = \frac{\sin[(n+1/2+\alpha_n/2)t - \alpha_n\pi/2]}{A_n^{\alpha_n}(2\sin(t/2))^{1+\alpha_n}}$$

and

$$r_n^{\alpha_n}(t) = -Im\left\{\frac{e^{-i\frac{t}{2}}}{2A_n^{\alpha_n}\sin\frac{t}{2}}\sum_{j=1}^{3}\frac{A_n^{\alpha_n-j}}{(1-e^{-it})^j} + \frac{e^{i(n+1/2)t}}{2A_n^{\alpha_n}\sin\frac{t}{2}}\frac{\sum\limits_{\nu=n+1}^{\infty}A_\nu^{\alpha_n-4}e^{-i\nu t}}{(1-e^{-it})^3}\right\} =$$

$$-\frac{1}{A_n^{\alpha_n}}Im\left\{\sum_{j=1}^{3} i^{-j}\left(2\sin\frac{t}{2}\right)^{-j-1}e^{i(j-1)t/2}A_n^{\alpha_n-j} + i\sum_{\nu=n+1}^{\infty}A_\nu^{\alpha_n-4}\left(2\sin\frac{t}{2}\right)^{-4}e^{i(n+2-\nu)t}\right\}. \tag{2.2}$$

Lemma 2.4. *For every natural n and $\alpha_n \in (-1, 1)$*

$$\left|[r_n^{\alpha_n}(t)]'\right| \leq \frac{C}{(1+\alpha_n)\,nt^3}.$$

Proof. Using representation (2.2) we get

$$[r_n^{\alpha_n}(t)]' = \frac{1}{A_n^{\alpha_n}}Im\left\{\sum_{j=1}^{3} i^{-j}\left(2\sin\frac{t}{2}\right)^{-j-2}(j+1)\cos\frac{t}{2}e^{i(j-1)t/2}A_n^{\alpha_n-j} - \frac{1}{2}\sum_{j=1}^{3} i^{-j+1}\left(2\sin\frac{t}{2}\right)^{-j-1}(j-1)e^{i(j-1)t/2}A_n^{\alpha_n-j} - 4i\sum_{j=n+1}^{\infty}A_\nu^{\alpha_n-4}\left(2\sin\frac{t}{2}\right)^{-5}\cos\frac{t}{2}e^{i(n+2-\nu)t} - \sum_{\nu=n+1}^{\infty}A_\nu^{\alpha_n-4}\left(2\sin\frac{t}{2}\right)^{-4}(n+2-\nu)e^{i(n+2-\nu)t}\right\} =: \sum_{k=1}^{4}N_k. \tag{2.3}$$

[2] In what follows by $C_1(i), C_2(i), C, ...$ we denote positive constants, respectively, absolute or dependent on parameters and indices which are, in general different in different formulas.

By Lemma 2.2 it is easy verify that for $t \in [\pi/n, \pi]$

$$N_1 = O\left(\frac{1}{1+\alpha_n}\sum_{j=1}^{3}(nt)^{-j}t^{-2}\right) = O\left(\frac{1}{(1+\alpha_n)nt^3}\right), \tag{2.4}$$

(2.5)

$$N_2 = O\left\{\sum_{j=1}^{3}\frac{1}{t^{1+j}}\frac{A_n^{\alpha_n-j}}{A_n^{\alpha_n}}\right\} = O\left\{\frac{1}{1+\alpha_n}\sum_{j=1}^{3}\frac{1}{(nt)^j t}\right\} = O\left(\frac{1}{(1+\alpha_n)nt^3}\right),$$

(2.6)

$$N_3 = O\left\{\frac{1}{A_n^{\alpha_n}}\sum_{\nu=n+1}^{\infty}\frac{\nu^{\alpha_n-4}}{t^5}\right\} = O\left\{\frac{1}{A_n^{\alpha_n}t^5}(n+1)^{\alpha_n-3}\right\} = O\left(\frac{1}{(1+\alpha_n)nt^3}\right).$$

Furthermore, it is easy to see that

$$N_4 = O\left\{\frac{1}{A_n^{\alpha_n}}\sum_{\nu=n+1}^{\infty}\frac{\nu^{\alpha_n-4}|n+2-\nu|}{t^4}\right\} = \tag{2.7}$$

$$O\left\{\frac{1}{(1+\alpha_n)(nt)^4} + \frac{n^{\alpha_n-4}}{A_n^{\alpha_n}t^4}\sum_{\nu=n+3}^{2n}(\nu-n-2) + \frac{1}{A_n^{\alpha_n}t^4}\sum_{\nu=2n+1}^{\infty}\nu^{\alpha_n-3}\right\} =$$

$$O\left\{\frac{1}{(1+\alpha_n)(nt)^4} + \frac{n^{\alpha_n-2}}{A_n^{\alpha_n}t^4} + \frac{n^{\alpha_n-2}}{A_n^{\alpha_n}t^4}\right\} =$$

$$O\left(\frac{1}{(1+\alpha_n)n^2t^4}\right) = O\left(\frac{1}{(1+\alpha_n)nt^3}\right).$$

Therefore, according to (2.3) - (2.7) the lemma follows. □

3. Proofs of the results

Proof of Theorem 1.5. Let $-1 < \alpha_n < 1, n = 1, 2, 3, \ldots$. We have (see [18, Ch. III, (5.4)])

(3.1)

$$\sigma_n^{-\alpha_n}(x,f) - f(x) = \frac{1}{\pi}\int_0^{\pi/n}\varphi(x,t)K_n^{-\alpha_n}(t)dt + \frac{1}{\pi}\int_{\pi/n}^{\pi}\varphi(x,t)K_n^{-\alpha_n}(t)dt =: I_1 + I_2,$$

where $\varphi(x,t)$ is defined by (1.3). Using Lemmas 2.1 and 2.3, by the formula for integration by parts, we get

$$|I_1| = \left|\frac{1}{\pi}\left[\Phi(x,t)K_n^{-\alpha_n}(t)\right]\Big|_0^{\pi/n} - \frac{1}{\pi}\int_0^{\pi/n}\Phi(x,t)\left[K_n^{\alpha_n}(x)\right]' dt\right| \leq \tag{3.2}$$

$$\frac{1}{\pi}\left|\Phi\left(x,\frac{\pi}{n}\right)\right|\left|K_n^{-\alpha_n}\left(\frac{\pi}{n}\right)\right| + \frac{1}{\pi}\int_0^{\pi/n}\sup_{0\leq t\leq\pi/n}|\Phi(x,t)|\left|\left[K_n^{\alpha_n}(x)\right]'\right| dt \leq$$

$$\frac{1}{\pi}\overline{\Phi}\left(x,\frac{\pi}{n}\right)\left(\frac{n}{1-\alpha_n} + \frac{1}{2} + \frac{\pi}{n}\cdot\frac{4n^2}{1-\alpha_n}\right) < \frac{5n}{1-\alpha_n}\overline{\Phi}\left(x,\frac{\pi}{n}\right).$$

On the other hand by the well-known representation [18, Ch. III, (5.14)] we have

$$I_2 = \frac{1}{\pi A_n^{-\alpha_n}} \int\limits_{\pi/n}^{\pi} \varphi(x,t) \frac{\sin\left[(n+1/2-\alpha_n/2)\,t+\alpha_n\pi/2\right]}{\left(2\sin\frac{t}{2}\right)^{1-\alpha_n}} dt + \tag{3.3}$$

$$\frac{1}{\pi} \int\limits_{\pi/n}^{\pi} \varphi(x,t) r_n^{-\alpha_n}(t) dt =: I_2^{(1)} + I_2^{(2)},$$

where

$$r_n^{-\alpha_n}(t) = -\frac{\alpha_n}{n} \cdot \frac{\theta_n(t)}{\left(2\sin\frac{t}{2}\right)^2}, \; |\theta_n(t)| \le 1. \tag{3.4}$$

Besides,

$$\left|I_2^{(2)}\right| = \left| \frac{1}{\pi} \left[\Phi(x,t) r_n^{-\alpha_n}(t)\right] \Big|_{\pi/n}^{\pi} - \frac{1}{\pi} \int\limits_{\pi/n}^{\pi} \Phi(x,t) \left[r_n^{-\alpha_n}(x)\right]' dt \right| \le$$

$$\frac{1}{\pi n} \left|\Phi\left(x,\pi\right)\right| + \frac{1}{\pi} \left|\Phi\left(x, \frac{\pi}{n}\right)\right| \left|r_n^{-\alpha_n}\left(\frac{\pi}{n}\right)\right| + \frac{C}{\pi\left(1-\alpha_n\right)n} \int\limits_{\pi/n}^{\pi} \frac{|\Phi(x,t)|}{t^3} dt.$$

Therefore, by (3.4) and (1.7) we can conclude that for the estimation I_2 it suffices (see (3.3)) to consider

$$M =: \frac{n^{\alpha_n}}{1-\alpha_n} \int\limits_{\pi/n}^{\pi} \varphi(x,t) g_n(t) cosntdt,$$

where

$$g_n(t) = \cos\frac{1-\alpha_n}{2} t \left(\sin\frac{t}{2}\right)^{\alpha_n - 1}. \tag{3.5}$$

It may be easily verified that

$$2M = \frac{n^{\alpha_n}}{1-\alpha_n} \left\{ \int\limits_{\pi/n}^{\pi-\pi/n} \left[\varphi(x,t) - \varphi(x,t+\pi/n)\right] g_n(t) \cos ntdt + \right. \tag{3.6}$$

$$\int\limits_{\pi/n}^{\pi-\pi/n} \varphi(x,t+\pi/n) \left[g_n(t) - g_n(t+\pi/n)\right] \cos ntdt -$$

$$\left. \int\limits_{0}^{\pi/n} \varphi(x,t+\pi/n) g_n(t+\pi/n) \cos ntdt + \int\limits_{\pi-\pi/n}^{\pi} \varphi(x,t) g_n(t) \cos ntdt \right\} =: \sum_{i=1}^{4} M_i.$$

By the condition (1.8) of Theorem 1.5 we obtain

$$M_1 = o(1), \; n \to \infty. \tag{3.7}$$

Now from (3.6), taking into account the estimation

$$|g_n(t+\pi/n)-g_n(t)| \le \frac{C(1-\alpha_n)}{nt^{2-\alpha_n}} \tag{3.8}$$

and condition (1.8), we get

$$\left| M_2 - \frac{n^{\alpha_n}}{1-\alpha_n}\int\limits_{\pi/n}^{\pi-\pi/n} \varphi(x,t)\left[g_n(t)-g_n(t+\pi/n)\right]\cos ntdt \right| \le \tag{3.9}$$

$$\frac{n^{\alpha_n}}{1-\alpha_n}\int\limits_{\pi/n}^{\pi-\pi/n} |\varphi(x,t+\pi/n)-\varphi(x,t)|\,|g_n(t)-g_n(t+\pi/n)|\,dt \le$$

$$Cn^{\alpha_n-1}\int\limits_{\pi/n}^{\pi-\pi/n} |\varphi(x,t+\pi/n)-\varphi(x,t)|\,\frac{dt}{t^{2-\alpha_n}} \le$$

$$Cn^{\alpha_n}\int\limits_{\pi/n}^{\pi-\pi/n} |\varphi(x,t+\pi/n)-\varphi(x,t)|\,\frac{dt}{t^{1-\alpha_n}} = o(1),\ \ n\to\infty.$$

Therefore, instead of M_2 it suffices to estimate

$$M_2^{\star} = \frac{n^{\alpha_n}}{1-\alpha_n}\int\limits_{\pi/n}^{\pi-\pi/n} \varphi(x,t)\left[g_n(t)-g_n(t+\pi/n)\right]\cos ntdt.$$

Analogously to the representation of (3.6) we have

$$2M_2^{\star} = \frac{n^{\alpha_n}}{1-\alpha_n}\left\{ \int\limits_{\pi/n}^{2\pi/n} \varphi(x,t)\left[g_n(t)-g_n(t+\pi/n)\right]\cos ntdt + \right. \tag{3.10}$$

$$\int\limits_{2\pi/n}^{\pi-2\pi/n} \left[\varphi(x,t)-\varphi(x,t+\pi/n)\right]\left[g_n(t)-g_n(t+\pi/n)\right]\cos ntdt +$$

$$\int\limits_{2\pi/n}^{\pi-2\pi/n} \varphi(x,t+\pi/n)\left[g_n(t)-2g_n(t+\pi/n)+g_n(t+2\pi/n)\right]\cos ntdt -$$

$$\int\limits_{0}^{2\pi/n} \varphi(x,t+\pi/n)\left[g_n(t+\pi/n)-g_n(t+2\pi/n)\right]\cos ntdt +$$

$$\left. \int\limits_{\pi-2\pi/n}^{\pi-\pi/n} \varphi(x,t)\left[g_n(t)-g_n(t+\pi/n)\right]\cos ntdt \right\} =: \sum_{i=5}^{9} M_i.$$

It is easy verify that for $t\in[\pi/n,\pi]$

$$|g_n''(t)| \le \frac{C(1-\alpha_n)}{t^{3-\alpha_n}}$$

and

$$\left|g_n^{(3)}(t)\right| \le \frac{C(1-\alpha_n)}{t^{4-\alpha_n}}.$$

Hence, using Lagrange theorem repeatedly, we obtain

$$|g_n'(t) - g_n'(t+\pi/n)| \le \frac{C(1-\alpha_n)}{nt^{3-\alpha_n}}, \tag{3.11}$$

$$|g_n(t) - 2g_n(t+\pi/n) + g_n(t+2\pi/n)| \le \frac{C(1-\alpha_n)}{n^2t^{3-\alpha_n}}, \tag{3.12}$$

$$|g_n'(t) - 2g_n'(t+\pi/n) + g_n'(t+2\pi/n)| \le \frac{C(1-\alpha_n)}{n^2t^{4-\alpha_n}}. \tag{3.13}$$

Now applying formula for integration by parts and take into account piece wise monotonicity of $\cos n\tau$ on $[0, \frac{2\pi}{n}]$ it follows by (1.7), (3.8), (3.11)

$$|M_5| \le \frac{n^{\alpha_n}}{1-\alpha_n}\left\{\left|g_n\left(\frac{2\pi}{n}\right) - g_n\left(\frac{3\pi}{n}\right)\right|\left|\int\limits_0^{2\pi/n}\varphi(x,t)\cos ntdt\right| + \right. \tag{3.14}$$

$$\left|g_n\left(\frac{\pi}{n}\right) - g_n\left(\frac{2\pi}{n}\right)\right|\left|\int\limits_0^{\pi/n}\varphi(x,t)\cos ntdt\right| +$$

$$\left|\int\limits_{\pi/n}^{2\pi/n}\int\limits_0^t \varphi(x,\tau)\cos n\tau d\tau\cdot\left[g_n'(t) - g_n'(t+\pi/n)\right]dt\right| \le$$

$$\frac{Cn^{\alpha_n}}{1-\alpha_n}\cdot\frac{(1-\alpha_n)\,n^{2-\alpha_n}}{n}\overline{\Phi}\left(x,\frac{2\pi}{n}\right) +$$

$$\frac{Cn^{\alpha_n}}{1-\alpha_n}\cdot\frac{1-\alpha_n}{n}\int\limits_{\pi/n}^{2\pi/n}\frac{\overline{\Phi}\,(x,t)}{t^{3-\alpha_n}}dt = o(1),\ n\to\infty.$$

On the other hand, by (3.8) and (1.8) we can conclude

$$|M_6| \le \frac{Cn^{\alpha_n}}{1-\alpha_n}\cdot\frac{1-\alpha_n}{n}\int\limits_{2\pi/n}^{\pi-2\pi/n}\frac{|\varphi(x,t) - \varphi\,(x,t+2\pi/n)|}{t^{2-\alpha_n}}dt \le \tag{3.15}$$

$$Cn^{\alpha_n}\int\limits_{2\pi/n}^{\pi-2\pi/n}\frac{|\varphi(x,t) - \varphi\,(x,t+2\pi/n)|}{t^{1-\alpha_n}}dt = o(1),\ n\to\infty.$$

Let

$$b_n(t) =: g_n(t) - 2g_n\left(t+\frac{\pi}{n}\right) + g_n\left(t+\frac{2\pi}{n}\right).$$

Then

$$|M_7| = \frac{n^{\alpha_n}}{1-\alpha_n}\left[b_n\left(\pi-\frac{2\pi}{n}\right)\cos(\pi n - 2\pi)\int\limits_0^{\pi-2\pi/n}\varphi\left(x,\tau+\frac{\pi}{n}\right)d\tau - \right.$$

$$b_n\left(\frac{2\pi}{n}\right)\cos(2\pi)\int\limits_0^{2\pi/n}\varphi\left(x,\tau+\frac{\pi}{n}\right)d\tau-$$

$$\left.\int\limits_{2\pi/n}^{\pi-2\pi/n}\int\limits_0^t\varphi\left(x,\tau+\frac{\pi}{n}\right)d\tau\left(b_n'(t)\cos nt-nb_n(t)\sin nt\right)dt\right].$$

Now taking into account (3.12), (3.13) and (1.7), we get

(3.16)

$$|M_7|\le\frac{Cn^{\alpha_n}}{1-\alpha_n}\left[\frac{(1-\alpha_n)}{n^2\left(\pi-\frac{2\pi}{n}\right)^{3-\alpha_n}}\left|\int\limits_{\pi/n}^{\pi-\pi/n}\varphi(x,\tau)\,d\tau\right|+\frac{1-\alpha_n}{n^2n^{\alpha_n-3}}\left|\int\limits_{\pi/n}^{3\pi/n}\varphi(x,\tau)\,d\tau\right|+\right.$$

$$\left.\int\limits_{2\pi/n}^{\pi-2\pi/n}\left|\int\limits_{\pi/n}^{t+\pi/n}\varphi(x,\tau)\,d\tau\right|\cdot\left(\frac{1-\alpha_n}{n^2t^{4-\alpha_n}}+\frac{n(1-\alpha_n)}{n^2t^{3-\alpha_n}}\right)dt\right]=o(1),\ n\to\infty.$$

In the same manner we can see that

$$M_8=o(1),\ n\to\infty. \tag{3.17}$$

Besides, by (3.8) simply obtain

$$M_9=o(1),\ n\to\infty. \tag{3.18}$$

Now applying (3.10), (3.14) - (3.18) we get that

$$M_2=o(1),\ n\to\infty. \tag{3.19}$$

On the other hand, for M_3 (see (3.6)) as well easily we have

$$|M_3|\le\frac{Cn^{\alpha_n}}{1-\alpha_n}\int\limits_0^{\pi/n}\overline{\Phi}\left(x,\frac{2\pi}{n}\right)n\cdot n^{1-\alpha_n}dt=o(1),\ n\to\infty. \tag{3.20}$$

Furthermore, since $g_n(t)$ is a decreasing function, by the second mean-value theorem it follows $\left(\xi_n\in\left(\pi-\frac{\pi}{n},\pi\right)\right)$ (see (3.6))

$$|M_4|=\left|\frac{n^{\alpha_n}}{1-\alpha_n}\int\limits_{\pi-\pi/n}^{\pi}\varphi(x,t)\,g_n(t)dt\right|=\left|\frac{n^{\alpha_n}}{1-\alpha_n}g_n\left(\pi-\frac{\pi}{n}\right)\int\limits_{\pi-\pi/n}^{\xi_n}\varphi(x,t)\,dt\right|\le$$

$$\frac{Cn^{\alpha_n}}{1-\alpha_n}\left|\int\limits_{2\pi/n}^{\xi_n}\varphi(x,t)\,dt-\int\limits_{2\pi/n}^{\pi-\pi/n}\varphi(x,t)\,dt\right|=$$

$$\frac{Cn^{\alpha_n}}{1-\alpha_n}\left|\int\limits_{\pi/n+(\pi-\xi_n)}^{\pi-\pi/n}\varphi\left(x,t+\xi_n+\frac{\pi}{n}-\pi\right)dt-\int\limits_{2\pi/n}^{\pi-\pi/n}\varphi(x,t)\,dt\right|\le$$

$$\frac{Cn^{\alpha_n}}{1-\alpha_n}\left\{\int\limits_{2\pi/n}^{\pi-\pi/n}\left|\varphi\left(x,t+\xi_n+\frac{\pi}{n}-\pi\right)-\varphi\left(x,t\right)\right|dt+\left|\int\limits_{2\pi/n}^{\pi/n+\pi-\xi_n}\varphi\left(x,t\right)dt\right|\right\}.$$

Hence according to the conditions of Theorem 1.5 we obtain

$$M_4 = o(1),\ \ n\to\infty. \tag{3.21}$$

Finally, on the base of (3.1), (3.2), (3.6), (3.7), (3.9), (3.10), (3.14) - (3.21) the proof of the first part of Theorem 1 is complete. It is easy see that in corresponding restrictions uniform (C,α_n)-summability of trigonometric Fourier series can be proved similarly. □

Proof of Corollary 1.2. By definition of $\overline{\Phi}(x,t)$ there exists a $t_0\in[0,t]$ such that

$$\overline{\Phi}(x,t)=\left|\int\limits_0^{t_0}\varphi(x,u)du\right|.$$

Thus

$$\frac{1}{t}\overline{\Phi}(x,t)\le\frac{1}{t_0}\left|\int\limits_0^{t_0}\varphi(x,u)du\right|.$$

Hence (1.4) implies

$$\overline{\Phi}(x,t)=o(t),\ \ t\to+0,$$

and for constant sequence α_n $(\alpha_n=\alpha\in(-1,0])$ this in turn implies (1.7).

Let $\alpha\in(-1,0]$ and $h\in(0,\pi/n]$. There exists $h_0\in(0,\pi/n]$ such that

$$\sup_{0<h\le\pi/n}\int\limits_{\pi/n}^{\pi}t^{-1-\alpha}|\varphi(x,t)-\varphi(x,t+h)|dt=$$

$$\int\limits_{\pi/n}^{\pi}t^{-1-\alpha}|\varphi(x,t)-\varphi(x,t+h_0)|dt.$$

We have

$$\left(\frac{\pi}{n}\right)^{\alpha}\sup_{0<h\le\frac{\pi}{n}}\int\limits_{\frac{\pi}{n}}^{\pi}t^{-1-\alpha}\left|\varphi(x,t)-\varphi(x,t+h)\right|dt\le$$

$$h_0^{\alpha}\int\limits_{h_0}^{\pi}t^{-1-\alpha}\left|\varphi(x,t)-\varphi(x,t+h_0)\right|dt.$$

Hence (1.6) implies (1.8). □

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