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Polarization of the Conformal Vacuum in the Milne Universe

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Abstract: The vacuum expectation values of the field squared and energy-momentum tensor for a massless scalar field are investigated in the Milne universe with general number of spatial dimensions. The vacuum state depends on the choice of the mode functions in the canonical quantization procedure and we assume that the field is prepared in the conformal vacuum. As the first step an integral representation for the difference of the Wightman functions corresponding to the conformal and Minkowski vacua is derived. The mean field squared and energy-momentum tensor are obtained in the coincidence limit. It is shown that the Minkowski vacuum state is interpreted as a thermal one with respect to the conformal vacuum. The thermal factor is of the Bose-Einstein type in odd dimensional space and of the Fermi-Dirac type in even number of spatial dimensions.

Keywords: vacuum polarization, Milne universe, conformal vacuum

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1. Introduction

It is well-known that the vacuum state in quantum field theory, in general, depends on the choice of the mode functions used in the canonical quantization procedure. If the Bogoliubov β -coefficient relating two sets of modes is different from zero then the corresponding vacuum states are inequivalent. An example, widely considered in the literature (see [1-3] and references therein), is provided by the Minkowski and Fulling-Rindler vacua in flat spacetime. The Minkowski vacuum corresponds to the modes of inertial observer and it preserves all the symmetries of the Minkowski spacetime (is maximally symmetric). The Fulling-Rindler vacuum is realized in the quantization procedure based on the modes of uniformly accelerated observers. For those observers horizons are present corresponding to the light cones that divide the spacetime into four sections. The right and left patches are referred to as the right and left Rindler wedges, respectively. The upper and lower patches are covered by the Milne coordinates and are referred to as the Milne universe.

The line element of the flat spacetime in the Milne coordinates is of the Friedmann-Robertson-Walker type with a scale factor being a linear function of the corresponding time coordinate. The spacetime is foliated by negative curvature spatial sections. The corresponding geometry serves as a simple model for the investigation of quantum-field-theoretical effects in time-dependent backgrounds. A relatively large number of problems are exactly solvable and motivated by that various aspects of the dynamics of quantum fields in the Milne universe have been discussed in [4-13]. In the present paper we consider the local characteristics of the conformal vacuum state (referred to as the C-vacuum) for a massless scalar field in the Milne universe.

The paper is organized as follows. In the next section we present the normal modes and the Wightman functions for the C- and Minkowski vacua in the case of a massive scalar field with general curvature coupling parameter. The general results are specified for a massless field in section 3. The expression for the mean field squared and energy-momentum tensor are discussed in section 4. The main results are summarized in section 6.

2. Wightman functions in conformal and Minkowski vacua

The line element for the $(D+1)$-dimensional Milne universe is expressed as

$$ds^2 = dt^2 - t^2(dr^2 + \sinh^2 r d\Omega_{D-1}^2), \tag{1}$$

where for the time and dimensionless radial coordinates one has $0 \le t < \infty$, $0 \le r < \infty$, and $d\Omega_{D-1}^2$ is the line element on a $(D-1)$-dimensional sphere. The spatial part corresponds to a constant negative curvature space covered by the hyperspherical coordinates (r, ϑ, ϕ). For the set of angular coordinates we have $\vartheta = (\theta_1, \theta_2, \ldots \theta_n)$, $0 \le \theta_k \le \pi$, $k = 1, 2, \ldots, n$ and $0 \le \phi \le 2\pi$, where $n = D-2$. Note that the spacetime described by the line element (1) is flat. That is explicitly seen introducing new coordinates $(T, \mathbf{R})$, with $\mathbf{R} = (R, \vartheta, \phi)$, in accordance with

$$T = t\cosh r, \; R = t\sinh r. \tag{2}$$

The line element (1) takes the Minkowskian form $ds^2 = dT^2 - dR^2 - R^2 d\Omega_{D-1}^2$ in hyperspherical spatial coordinates. As seen from (2), $T > R$ and the Milne coordinates (t, r, ϑ, ϕ) cover the patch of the Minkowski spacetime inside the future light cone. In the region corresponding to the past light cone we take $T = t\cosh r$, $R = -t\sinh r$ with the time coordinate $-\infty < t \le 0$. The remaining regions of the Minkowski spacetime, $R > |T|$, correspond to the Rindler patches.

Passing to new coordinates $(\eta, \bar{r})$, $-\infty < \eta < \infty$, in accordance with

$$t = ae^{\eta/a} \equiv ae^{\bar{\eta}}, \; r = \bar{r}/a, \tag{3}$$

where a is a constant with dimension of length, the line element (1) is presented in a conformally-static form

$$ds^2 = (t/a)^2 \left[d\eta^2 - d\bar{r}^2 - a^2 \sinh^2(\bar{r}/a) d\Omega_{D-1}^2 \right]. \tag{3}$$

The line element in the square brackets describes a static spacetime with a constant negative curvature space. The curvature radius of the latter is determined by a and in the limit $a \to \infty$ for fixed η and $\bar{r}$, from (4) the Minkowskian line element in spherical coordinates is obtained. For two spacetime points $(T', \mathbf{R}')$ and $(T, \mathbf{R})$ the spacetime interval between them is expressed as

$$(\Delta T)^2 - |\Delta\mathbf{R}|^2 = 2tt'(\cosh\Delta\bar{\eta} - \cosh\zeta), \tag{4}$$

where $\Delta T = T' - T$, $\Delta\mathbf{R} = \mathbf{R}' - \mathbf{R}$, $\Delta\bar{\eta} = \bar{\eta}' - \bar{\eta}$, and ζ is defined by the relation

$$\bar{u} = \cosh r \cosh r' - \sinh r \sinh r' \cos\theta = \cosh\zeta. \tag{5}$$

Here, θ is the angle between directions (ϑ', ϕ') and (ϑ, ϕ). In the special case $\theta = 0$, corresponding to points on the same radial directions, we have $\zeta = r' - r \equiv \Delta r$.

Having described the background geometry we turn to the field content. We consider a quantum scalar field $\varphi(x)$ with curvature coupling parameter ξ. The field equation reads

$$\left(\Box + m^2 + \xi R\right)\varphi = 0, \tag{6}$$

where $\Box$ is the d'Alembert operator and for background under consideration the Ricci scalar is zero, $R = 0$. Let $\{\varphi_\sigma(x), \varphi_\sigma^*(x)\}$ be the complete set of mode functions obeying the field equation and specified by the quantum numbers σ. For the modes corresponding to the C-vacuum $\sigma = (z, m_p)$ and

$$\varphi_\sigma\left(x\right) = N_\sigma \frac{J_{-iz}(mt)}{t^{(D-1)/2}} \frac{P_{iz-1/2}^{1-D/2-l}(\cosh r)}{\sinh^{D/2-1} r} Y(m_p; \vartheta, \phi), \tag{7}$$

where $J_\nu\left(x\right)$ is the Bessel function, $P_\rho^\gamma(x)$ is the associated Legendre function of the first kind and $Y(m_p; \vartheta, \phi)$ are the hyperspherical harmonics. In (8), $0 \le z < \infty$ and $m_p = (l, m_1, \dots, m_n)$ with $l = 0, 1, 2, \dots$. For the integers $m_1, m_2, \dots, m_n$ one has $-m_{n-1} \le m_n \le m_{n-1}$ and $0 \le m_{n-1} \le m_{n-2} \le \cdots \le m_1 \le l$. The coefficient N_σ is determined by the standard normalization condition for the equation (7) and is determined from

$$\left|N_\sigma\right|^2 = \frac{z\left|\Gamma\left((D-1)/2 + l + iz\right)\right|^2}{2N\left(m_p\right)}. \tag{8}$$

The factor $N\left(m_p\right)$ comes from the normalization condition for the hyperspherical harmonics and its explicit form will not be required in the discussion below.

The correlations of the vacuum fluctuations of quantum fields at different spacetime points x and x' are determined by the two-point functions. Here we consider the Wightman function. The latter can be evaluated by using the mode-sum formula

$$W(x, x') = \sum_\sigma \varphi_\sigma\left(x\right)\varphi_\sigma^*\left(x'\right). \tag{9}$$

The expression of the Wightman function for the conformal vacuum (denoted here as $W_{\mathrm{C}}(x, x')$) can be obtained by using the corresponding formula for the Hadamard function from [13]:

$$W_{\mathrm{C}}(x, x') = \frac{\left(tt'\right)^{(1-D)/2}}{2nS_D} \sum_{l=0}^{\infty} \frac{\left(2l+n\right)C_l^{n/2}(\cos\theta)}{(\sinh r \sinh r')^{D/2-1}} \int_0^\infty dy\, y \,|\, \Gamma(l + \left(D-1\right)/2 + iy) \,|^2 \\ \times J_{-iy}(mt) J_{iy}(mt') P_{iy-1/2}^{-\mu}(u) P_{iy-1/2}^{-\mu}(u'), \tag{10}$$

where $S_D = 2\pi^{D/2}/\Gamma\left(D/2\right)$, $C_l^{n/2}(\cos\theta)$ is the Gegenbauer polynomial, and

$$u = \cosh r, u' = \cosh r'. \tag{11}$$

By using the addition theorem for the associated Legendre functions from [14] (for the correction of the missprint see [19]) it can be shown that

$$\sum_{l=0}^{\infty}\left(l+\frac{n}{2}\right)C_l^{n/2}\left(\cos\theta\right)\left|\Gamma\left(l+\left(D-1\right)/2+iy\right)\right|^2 P_{iy-1/2}^{-l-n/2}\left(u\right)P_{iy-1/2}^{-l-n/2}\left(u'\right)$$
$$=\frac{2^{-n/2}}{\Gamma\left(n/2\right)}\left[\left(u^2-1\right)\left(u'^2-1\right)\right]^{n/4}\left|\Gamma\left(\left(D-1\right)/2+iy\right)\right|^2\frac{P_{iy-1/2}^{-n/2}\left(\bar{u}\right)}{\left(\bar{u}^2-1\right)^{n/4}}, \quad (12)$$

with $\bar{u}$ given by (6). By using this result, the expression (11) is further simplified as

$$W_{\mathrm{C}}(x,x')=\frac{\left(tt'\right)^{(1-D)/2}}{2\left(2\pi\right)^{D/2}}\int_0^{\infty}dy\,yJ_{-iy}(mt)J_{iy}(mt')\left|\Gamma\left(iy+\left(D-1\right)/2\right)\right|^2\frac{P_{iy-1/2}^{-n/2}\left(\bar{u}\right)}{\left(\bar{u}^2-1\right)^{n/4}}. \quad (13)$$

The VEVs of the field squared and energy-momentum tensor are obtained from two-point functions in the coincidence limit. That limit is divergent and a renormalization is required. In the present paper we are interested in the difference of the local characteristics of the C- and Minkowski vacua. The latter is obtained from the difference $\Delta W(x,x')=W_{\mathrm{C}}(x,x')-W_{\mathrm{M}}(x,x')$, where

$$W_{\mathrm{M}}(x,x')=\frac{m^{(D-1)/2}}{\left(2\pi\right)^{(D+1)/2}}\frac{K_{(D-1)/2}(m\sqrt{\left|\Delta\mathbf{R}\right|^2-\left(\Delta T\right)^2})}{\left[\left|\Delta\mathbf{R}\right|^2-\left(\Delta T\right)^2\right]^{(D-1)/4}}, \quad (14)$$

is the Wightman function for the Minkowski vacuum. Here, we assume that $\left|\Delta\mathbf{R}\right|>|\Delta T|$. Note that, in accordance with (5), this corresponds to $|\zeta|>|\Delta\bar{\eta}|$. The expressions in the other regions of the Minkowski spacetime are obtained by the analytical continuation. The local geometry for both the C-vacuum in the Milne patch and for the Minkowski vacuum is the same and, hence, the difference $\Delta W_{\mathrm{C}}(x,x')$ is finite and can be directly used for the evaluation of the local VEVs.

3. Wightman function for a massless field

The expression (14) for the Wightman function is further simplified for a massless field. In the limit $m\to 0$ for the product of the Bessel functions one has

$$J_{-iy}(mt)J_{iy}(mt')\approx\frac{\sinh\left(\pi y\right)}{\pi y}e^{iy\Delta\bar{\eta}}, \quad (15)$$

and the formula takes the form

$$W_{\mathrm{C}}(x,x')=\frac{\left(tt'\right)^{(1-D)/2}}{\left(2\pi\right)^{D/2+1}}\int_0^{\infty}dy\sinh\left(\pi y\right)e^{iy\Delta\bar{\eta}}\left|\Gamma\left(iy+\frac{D-1}{2}\right)\right|^2\frac{P_{iy-1/2}^{-n/2}\left(\bar{u}\right)}{\left(\bar{u}^2-1\right)^{n/4}}. \quad (16)$$

By using the properties of the associated Legendre function this expression can be rewritten as

$$W_{\mathrm{C}}(x,x') = (-1)^q \frac{(tt')^{(1-D)/2}}{(2\pi)^{D/2+1}} \partial_{\bar{u}}^q \int_0^\infty dy \sinh(\pi y) e^{iy\Delta\bar{\eta}} \times \left| \Gamma\left(\frac{D-1}{2} - q + iy \right) \right|^2 \frac{P_{iy-1/2}^{1-D/2+q}(\bar{u})}{(\bar{u}^2-1)^{(D-2q-2)/4}}, \quad (17)$$

with q being a non-negative integer. In the massless limit, by taking into account the relation (5), for the Minkowski vacuum one gets

$$W_{\mathrm{M}}(x,x') = \frac{(tt')^{(1-D)/2}}{2(2\pi)^{(D+1)/2}} \frac{\Gamma((D-1)/2)}{(\cosh\zeta - \cosh\Delta\bar{\eta})^{(D-1)/2}}. \quad (18)$$

For the further transformation we will consider the odd and even values for the spatial dimension D separately.

For even values of spatial dimension, taking $q = D/2-1$, the Wightman function is expressed as

$$W_{\mathrm{C}}(x,x') = -\frac{(tt')^{(1-D)/2}}{2(-2\pi)^{D/2}} \left(\frac{\partial_\zeta}{\sinh\zeta} \right)^{D/2-1} \int_0^\infty dy \tanh(\pi y) e^{iy\Delta\bar{\eta}} P_{iy-1/2}(\cosh\zeta). \quad (19)$$

The corresponding function for the Minkowski vacuum is presented in the form

$$W_{\mathrm{M}}(x,x') = -\frac{(tt')^{(1-D)/2}}{2^{3/2}(-2\pi)^{D/2}} \left(\frac{\partial_\zeta}{\sinh\zeta} \right)^{D/2-1} \frac{1}{\sqrt{\cosh\zeta - \cosh\Delta\bar{\eta}}}. \quad (20)$$

For the evaluation of the difference of the Wightman functions it is convenient to present $W_{\mathrm{M}}(x,x')$ in an integral form

$$W_{\mathrm{M}}(x,x') = -\frac{(tt')^{(1-D)/2}}{2(-2\pi)^{D/2}} \left(\frac{\partial_\zeta}{\sinh\zeta} \right)^{D/2-1} \int_0^\infty dy \cos(y\Delta\bar{\eta}) P_{iy-1/2}(\cosh\zeta). \quad (21)$$

Here we have used the relation [16]

$$\int_0^\infty dy \cos(yv) P_{iy-1/2}(w) = \frac{1}{\sqrt{2}\sqrt{w-\cosh v}}, \quad (22)$$

valid for $w > \cosh v$. Thus, for the difference of the Wightman functions we get

$$\Delta W(x,x') = \frac{(tt')^{(1-D)/2}}{2(-2\pi)^{D/2}} \int_0^\infty dy \left[\frac{2\cos(y\Delta\bar{\eta})}{e^{2\pi y}+1} - i\tanh(\pi y)\sin(y\Delta\bar{\eta}) \right] \frac{P_{iy-1/2}^{D/2-1}(\bar{u})}{(\bar{u}^2-1)^{(D-2)/4}}, \quad (23)$$

where the relation $\partial_{\bar{u}}^{D/2-1} P_{iz-1/2}(\bar{u}) = (\bar{u}^2-1)^{(2-D)/4} P_{iz-1/2}^{D/2-1}(\bar{u})$ was used.

Now we pass to the odd values of D. For this case in (18) we take $q=(D-3)/2$ and the associated Legendre function in the integrand of (18) is reduced to the function $P_{iy-1/2}^{-1/2}(\bar{u})$. The latter is expressed in terms of elementary functions and for $(\Delta\bar{\eta})^2 \neq \zeta^2$ one gets

$$W_{\mathrm{C}}(x,x') = \frac{(tt')^{(1-D)/2}}{2(-2\pi)^{(D+1)/2}} \left(\frac{\partial_\zeta}{\sinh\zeta}\right)^{(D-3)/2} \frac{2\zeta/\sinh\zeta}{\zeta^2-(\Delta\bar{\eta})^2}, \tag{24}$$

where ζ is defined in accordance with (6). For a massless field and for odd values of D the Wightman function (19) for the Minkowski vacuum is presented as

$$W_{\mathrm{M}}(x,x') = \frac{(tt')^{(1-D)/2}}{2(-2\pi)^{(D+1)/2}} \left(\frac{\partial_\zeta}{\sinh\zeta}\right)^{(D-3)/2} \frac{1}{\cosh\zeta - \cosh\Delta\bar{\eta}}. \tag{25}$$

Here we have used the relation (5). For the difference of the Wightman functions, that determines the difference in the local VEVs, we obtain

$$\Delta W(x,x') = \frac{(tt')^{(1-D)/2}}{2(-2\pi)^{(D+1)/2}} \left(\frac{\partial_\zeta}{\sinh\zeta}\right)^{(D-3)/2} \left[\frac{2\zeta/\sinh\zeta}{\zeta^2-(\Delta\bar{\eta})^2} - \frac{1}{\cosh\zeta-\cosh\Delta\bar{\eta}}\right]. \tag{26}$$

This representation is well adapted for the evaluation of the differences between local VEVs of the field squared and energy-momentum tensor.

In the discussion above we have considered the difference in the Wightman functions for the C- and Minkowski vacua. Similar expressions are obtained for the differences of other two-point functions. In particular the VEVs of the field squared are obtained from the Hadamard function $G(x,x')$. For odd D the corresponding expression is obtained from (27) with an additional coefficient 2. In the case of even D from (24) for the difference $\Delta G(x,x') = G_{\mathrm{C}}(x,x') - G_{\mathrm{M}}(x,x')$ we get

$$\Delta G(x,x') = \frac{2(tt')^{(1-D)/2}}{(-2\pi)^{D/2}} \int_0^\infty dy \frac{\cos(y\Delta\bar{\eta})}{e^{2\pi y}+1} \frac{P_{iy-1/2}^{D/2-1}(\bar{u})}{(\bar{u}^2-1)^{(D-2)/4}}. \tag{27}$$

4. VEV of the field squared

As a local characteristic of the C-vacuum first let us consider the VEV of the field squared. It is obtained from the Hadamard function in the coincidence limit as $\Delta\langle\varphi^2\rangle = \lim_{x'\to x} \Delta G(x,x')/2$. If we renormalize the corresponding VEV for the Minkowski vacuum to zero, $\langle\varphi^2\rangle_{\mathrm{M}}^{(\mathrm{ren})} = 0$, then $\Delta\langle\varphi^2\rangle$ gives the renormalized VEV for the C-vacuum, $\Delta\langle\varphi^2\rangle = \langle\varphi^2\rangle_{\mathrm{C}}^{(\mathrm{ren})}$. For even values of D we use the relation (see, for example, [17])

$$\lim_{\bar{u}\to 1} \frac{P_{iy-1/2}^{D/2-1}(\bar{u})}{(\bar{u}^2-1)^{(D-2)/4}} = \frac{2^{1-D/2}\Gamma(iy+(D-1)/2)}{\Gamma(D/2)\Gamma(iy-(D-3)/2)}. \tag{28}$$

The expression for the VEV can be transformed to the form

$$\Delta\left\langle\varphi^2\right\rangle = -\frac{2^{-D}t^{1-D}}{\pi^{D/2+1}\Gamma(D/2)}\int_0^\infty dy e^{-\pi y}\left|\Gamma\left(iy+\left(D-1\right)/2\right)\right|^2. \quad (29)$$

Alternatively, by using the properties of the gamma function we can see that

$$\Delta\left\langle\varphi^2\right\rangle = -\frac{2(4\pi)^{-D/2}}{\Gamma(D/2)t^{D-1}}\int_0^\infty dy\frac{y^{D-2}A_D\left(y\right)}{e^{2\pi y}+1}, \quad (30)$$

where for even D we have defined

$$A_D\left(y\right) = \prod_{l=0}^{D/2-2}\left[\left(l+1/2\right)^2/y^2+1\right]. \quad (31)$$

As seen, the VEV is always negative.

For odd values of D it is convenient firstly to put $\Delta\overline{\eta}=0$, $\theta=0$ in the expression (27). With this choice we have $\zeta=\Delta r$. The VEV of the field squared is presented as

$$\Delta\left\langle\varphi^2\right\rangle = -\frac{b_D t^{1-D}}{12(2\pi)^{(D+1)/2}}, \quad (32)$$

where the coefficient b_D is defined by the relation

$$b_D = 6(-1)^{\frac{D-1}{2}}\lim_{\overline{u}\to 1}\partial_{\overline{u}}^{(D-3)/2}\left[\frac{2\left(\overline{u}^2-1\right)^{-1/2}}{\operatorname{arccosh}\left(\overline{u}\right)}-\frac{1}{\overline{u}-1}\right]. \quad (33)$$

In particular,

$$b_3=1, b_5=\frac{11}{30}, b_7=\frac{191}{630}, b_9=\frac{2497}{6300}. \quad (34)$$

It is of interest to note that the expression (33) can also be written in the form

$$\Delta\left\langle\varphi^2\right\rangle = -\frac{2(4\pi)^{-D/2}}{\Gamma(D/2)t^{D-1}}\int_0^\infty dy\frac{y^{D-2}A_D\left(y\right)}{e^{2\pi y}-1}, \quad (35)$$

where for odd D

$$A_D\left(y\right) = \prod_{l=0}^{(D-3)/2}\left(l^2/y^2+1\right). \quad (36)$$

As before, the VEV is negative. We can combine the expressions for even and odd values of the spatial dimension in a single formula

$$\Delta\left\langle\varphi^2\right\rangle = -\frac{B_D}{t^{D-1}}, \quad (37)$$

where

$$B_D = \frac{2(4\pi)^{-D/2}}{\Gamma(D/2)} \int_0^\infty dy \frac{y^{D-2} A_D(y)}{e^{2\pi y} + (-1)^D}. \tag{38}$$

5. VEV of the energy-momentum tensor

In this section we will consider the VEV of the energy-momentum tensor. Through the Einstein semiclassical equations it determines the backreaction of quantum effects on the background geometry. By taking into account that for the background under consideration the Ricci tensor is zero, the difference in the VEVs for the C- and Minkowski vacua, $\Delta\langle T_{ik}\rangle = \langle T_{ik}\rangle_{\mathrm{C}} - \langle T_{ik}\rangle_{\mathrm{M}}$, is evaluated on the basis of the formula

$$\Delta\langle T_{ik}\rangle = \frac{1}{2}\lim_{x'\to x} \partial_{i'}\partial_k \Delta G(x,x') + \left[\left(\xi - \frac{1}{4}\right) g_{ik}\Box - \xi \nabla_i \nabla_k\right] \Delta\langle\varphi^2\rangle, \tag{39}$$

where ∇_i is the covariant derivative operator and ξ is the curvature coupling parameter. If we renormalize the VEV for the Minkowski vacuum to zero, then $\Delta\langle T_{ik}\rangle$ gives the mean energy-momentum tensor for the C-vacuum. Relations between the separate components follow from the symmetry of the problem and from general relations. First of all, the spatial geometry is isotropic and the stresses are equal, $\Delta\langle T_1^1\rangle = \Delta\langle T_2^2\rangle = \cdots = \Delta\langle T_D^D\rangle$. Next, by taking into account that all the components depend on the time coordinate only, from the covariant continuity equation $\nabla_k \langle T_i^k\rangle = 0$ we get

$$\Delta\langle T_1^1\rangle = \frac{1}{Dt^{D-1}} \partial_t \left(t^D \Delta\langle T_0^0\rangle\right). \tag{40}$$

Additionally, one has the trace relation

$$\Delta\langle T_i^i\rangle = D(\xi - \xi_D)\Box\Delta\langle\varphi^2\rangle, \tag{41}$$

where $\xi_D = (D-1)/(4D)$ is the curvature coupling parameter for a conformally coupled field. By taking into account that $\Delta\langle\varphi^2\rangle \propto 1/t^{D-1}$ it is easy to see that $\Box\Delta\langle\varphi^2\rangle = 0$ and, hence, the VEV of the energy-momentum tensor is traceless. This leads to the relation $\Delta\langle T_0^0\rangle = -D\Delta\langle T_1^1\rangle$ between the energy density and the stresses. From this relation and from (41) we get

$$\Delta\langle T_i^k\rangle = \frac{C_D}{t^{D+1}} \operatorname{diag}(1, -1/D, \ldots, -1/D). \tag{42}$$

The problem is reduced to the evaluation of the constant C_D.

We will evaluate the component $\Delta\langle T_{11}\rangle$. For the derivative in the last term of (40) one obtains

$$\nabla_1 \nabla_1 \Delta\langle\varphi^2\rangle = (D-1)\Delta\langle\varphi^2\rangle. \tag{43}$$

For the evaluation of the first term we consider the cases of even and odd D separately. For even D the difference of the Hadamard functions is given by (28). Taking $\theta = 0$, $\Delta\eta = 0$, in the coincidence limit we get

$$\frac{1}{2}\lim_{x'\to x}\partial_{1'}\partial_1\Delta G\left(x,x'\right) = -\frac{t^{1-D}}{\left(-2\pi\right)^{D/2}}\int_0^\infty dy\frac{1}{e^{2\pi y}+1}\lim_{\bar{u}\to 1}\partial_{\Delta r}^2\frac{P_{iy-1/2}^{D/2-1}\left(\cosh\Delta r\right)}{\sinh^{D/2-1}\Delta r}. \tag{44}$$

From the recurrence relations for the associated Legendre function the following relation can be shown:

$$\partial_{\Delta r}^2\frac{P_{iy-1/2}^{D/2-1}\left(\cosh\Delta r\right)}{\sinh^{D/2-1}\Delta r} = \frac{P_{iy-1/2}^{D/2+1}\left(\cosh\Delta r\right)}{\sinh^{D/2-1}\Delta r} + \cosh\Delta r\frac{P_{iy-1/2}^{D/2}\left(\cosh\Delta r\right)}{\sinh^{D/2}\Delta r}. \tag{45}$$

The contribution of the first term in the right-hand side tends to zero in the limit $\Delta r \to 0$ and we get

$$\frac{1}{2}\lim_{x'\to x}\partial_{1'}\partial_1\Delta G\left(x,x'\right) = -\frac{2^{-D}\pi^{-D/2-1}}{D\Gamma(D/2)t^{D-1}}\int_0^\infty dy e^{-\pi y}\left|\Gamma\left(iy+\left(D+1\right)/2\right)\right|^2. \tag{46}$$

Substituting (44) and (47) into (40) with $i = k = 1$ and comparing with (43) for even values of D one finds

$$\begin{aligned} C_D &= -\frac{\pi^{-D/2-1}}{2^D\Gamma(D/2)}\int_0^\infty dy e^{-\pi y}\left|\Gamma\left(iy+\left(D-1\right)/2\right)\right|^2\left[y^2+D\left(D-1\right)\left(\xi_D-\xi\right)\right] \\ &= -\frac{2^{1-D}\pi^{-D/2}}{\Gamma(D/2)}\int_0^\infty dy\frac{y^{D-2}A_D\left(y\right)}{e^{2\pi y}+1}\left[y^2+D\left(D-1\right)\left(\xi_D-\xi\right)\right], \end{aligned} \tag{47}$$

where $A_D(y)$ is given by (32).

In the case of odd D, as in the case of the mean field squared, we can take $\Delta\bar{\eta} = 0$ and $\theta = 0$. For the coincidence limit of the derivative of the Hadamard function we get

$$\lim_{x'\to x}\partial_{1'}\partial_1\Delta G\left(x,x'\right) = \frac{t^{1-D}}{\left(-2\pi\right)^{(D+1)/2}}\lim_{r'\to r}\partial_{\Delta r}^2\partial_{\bar{u}}^{(D-3)/2}\left[\frac{1}{\bar{u}-1} - \frac{2\left(\bar{u}^2-1\right)^{-1/2}}{\operatorname{arccosh}\left(\bar{u}\right)}\right]. \tag{48}$$

By taking into account that $\bar{u} = \cosh\Delta r$, we see that $\partial_{\Delta r}^2 = \bar{u}\partial_{\bar{u}} + (\bar{u}^2-1)\partial_{\bar{u}}^2$. The contribution of the last term vanishes in the limit $r' \to r$ ($\bar{u} \to 1$) and by using the definition (34) one obtains

$$\lim_{x'\to x}\partial_{1'}\partial_1\Delta G\left(x,x'\right) = -\frac{b_{D+2}t^{1-D}}{6\left(2\pi\right)^{(D+1)/2}}. \tag{49}$$

This result with the combination of (33) and (44) leads to the following expression for the coefficient in (43):

$$C_D = D\frac{\xi\left(D-1\right)b_D - b_{D+2}}{12(2\pi)^{(D+1)/2}}. \tag{50}$$

It can be checked that the latter expression, valid for odd values of D, may also be written in the integral form

$$C_D = -\frac{2^{1-D}}{\pi^{D/2}\Gamma(D/2)}\int_0^\infty dy \frac{y^{D-2}A_D(y)}{e^{2\pi y}-1}\left[y^2 + D(D-1)(\xi_D - \xi)\right], \tag{51}$$

with $A_D(y)$ defined by (32).

Let us compare the differences $\Delta\langle\varphi^2\rangle$ and $\Delta\langle T_{ik}\rangle$ with the differences in the corresponding VEVs between the Fulling-Rindler and Minkowski vacua. The right Rindler wedge is covered by the coordinates $(\tau_{\rm R}, \chi, \mathbf{x}_{\rm R})$, with $-\infty < \tau_{\rm R} < +\infty$, $0 \le \chi < \infty$, $\mathbf{x}_{\rm R} = (x_{\rm R}^2, \ldots, x_{\rm R}^D)$, and the corresponding line element has the form

$$ds_{\rm M}^2 = \chi^2 d\tau_{\rm R}^2 - d\chi^2 - d\mathbf{x}_{\rm R}^2. \tag{52}$$

For a massless field, the difference of the mean field squared in the Fulling-Rindler and Minkowski vacua is given by the expression [18]

$$\langle\varphi^2\rangle_{\rm FR} - \langle\varphi^2\rangle_{\rm M} = -\frac{B_D}{\chi^{D-1}}, \tag{53}$$

where the coefficient B_D is the same as in (39). The difference in VEV of the energy-momentum tensor is expressed as (no summation over i)

$$\langle T_i^k\rangle_{\rm FR} - \langle T_i^k\rangle_{\rm M} = -\frac{2\delta_i^k(4\pi)^{-D/2}}{\Gamma(D/2)\chi^{D+1}}\int_0^\infty dy \frac{y^{D-2}A_D(y)}{e^{2\pi y}+(-1)^D} f_0^{(i)}(y), \tag{54}$$

where

$$\begin{aligned} f_0^{(0)}(y) &= -Df_0^{(1)}(y) = y^2 + D(D-1)(\xi_D - \xi), \\ f_0^{(i)}(y) &= -y^2/D + (D-1)^2(\xi_D - \xi), \; i = 2,3,\ldots,D. \end{aligned} \tag{55}$$

For a conformally coupled massless field one gets

$$\langle T_i^k\rangle_{\rm FR} - \langle T_i^k\rangle_{\rm M} = \frac{C_D}{\chi^{D+1}}{\rm diag}(1, -1/D, \ldots, -1/D), \tag{56}$$

with the same coefficient as in (43). For non-conformally coupled fields the stresses for the Fulling-Rindler vacuum are anisotropic.

6. Conclusions

The Milne universe is well suited for studying various aspects of the influence of background geometry on the properties of quantum fields. In particular, the study of properties of vacuum state is of special interest. We have investigated the local properties of the C-vacuum for a massless scalar field. Among the most important local characteristics are the expectation values of the field squared and energy-momentum tensor. They are obtained from the two-point functions in the coincidence limit of the arguments. For the renormalization of the VEVs the subtraction of the corresponding VEVs for the Minkowski vacuum is sufficient. This is related to the fact that the Milne universe is flat and the divergences in the VEVs for C- and Minkowski vacua are the same.

For a massless scalar field we have derived relatively simple representations for the difference in the Wightman and Hadamard functions for C- and Minkowski vacua. The renormalized mean field squared for C-vacuum is directly obtained from the difference taking the coincidence limit. For the evaluation of the renormalized mean energy-momentum tensor we have used the formula (40). The mean field squared is given by (38) with the coefficient (39). From the symmetry of the problem it follows that VEV of the energy-momentum tensor should have the form (43). In order to obtain the expression for the coefficient C_D we have evaluated the 11-component. The coefficient is given by (48) for even D and by (52) in odd spatial dimensions. The formulas for the mean field squared and energy-momentum tensor show that, from the point of view of the quantization procedure in terms of the mode functions based on the line element (1), the Minkowski vacuum appears as a thermal state. It is of interest to note that thermal factor is the Bose-Einstein one for odd number of spatial dimensions and Fermi-Dirac type in even number of spatial dimensions. We have emphasized that this feature is also present in the relations between the VEVs in Fulling-Rindler and Minkowski vacua. Similar features between the hyperbolic and Bunch-Davies vacua in de Sitter spacetime have been discussed in [15,19].

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