THE BUBNOV–GALERKIN PROCEDURE IN PROBLEMS OF MOBILE (SCANNING) CONTROL FOR SYSTEMS WITH DISTRIBUTED PARAMETERS

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Բուբնով-Գալյորկինի եղանակը բաշխված պարամետրերով համակարգերի շարժական դեկավարման խնդիրներում

Բաշխված պարամետրերով համակարգերի շարժական դեկավարման խնդիրների լուծման համար առաջարկվում է օգտագործել Բուբնով–Գալյորկինի հայտնի եղանակը: Առաջարկվող ալգորիթմը մանրամասն նկարագրված է ջերմահաղորդականության եռաչափ գծային հավասարման համար: Այն թույլ է տալիս դեկավարման խնդրի լուծումը հանգեցնելու վերջավոր չափանի ոչ գծային մոմենտների պրոբլեմի: Դրա ստացման եղանակը դիտարկվում է ջերմահաղորդականության միաչափ հավասարման համար: Կառուցվում է ստացված մոմենտների պրոբլեմի լուծումը մասնավոր դեպքում: Արդյունքների հիման վրա միաչափ դեպքում ձողի համար իրականացվել է համակարգչային նմանակում՝ COMSOL Multiphysics միջավայրում: Արտացոլվել են դեկավարման ֆունկցիայի հիմնական կախվածությունները խնդրի տվյալներից: Ձողի վիճակը ներկայացվում է գրաֆիկորեն՝ աղբյուրի ինտենսիվության մի քանի (հաստատուն) արժեքի դեպքում: Համեմատելու համար բերվում են համապատասխան արդյունքները՝ դեկավարման բացակայության (զրոյական հզորությամբ աղբյուր) դեպքում:

Թույլատրելի դեկավարումների ընդլայնված դասի համար նկարագրված է ստացված ոչ գծային հավասարումների համակարգի թվային արդյունավետ լուծման սխեմա: Դեկավարման պարամետրերի որոշումը հանգեցված է ոչ գծային ծրագրավորման պարզագույն խնդրի:

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Метод Бубнова–Галёркина в задачах подвижного управления системами с распределёнными параметрами

Для решения задач подвижного управления системами с распределёнными параметрами предлагаем применить метод Бубнова–Галёркина. Алгоритм подробно описан для трёхмерного линейного уравнения теплопроводности. Он позволяет свести решение задачи управления к конечномерной нелинейной проблеме моментов. Процедура получения проблемы моментов подробно демонстрируется на примере одномерного уравнения теплопроводности. Найдено решение полученной проблемы моментов в частном случае. На основе полученных результатов одномерная задача была симулирована в среде COMSOL Multiphysics. Выявлены основные зависимости функции управления от входных данных задачи. Графически отражены состояния стержня для разных (постоянных) интенсивностей источника. Для сравнения приведены соответствующие иллюстрации при отсутствии управления (источник нулевой интенсивности).

В случае расширенного класса допустимых управлений предложена эффективная схема численного решения полученной системы нелинейных ограничений. Определение параметров управления сводится к простейшей задаче нелинейного программирования.

We suggest to apply the Bubnov–Galerkin procedure to solve scanning control problems for systems with distributed parameters. The algorithm is described in details for three-dimensional linear heat equation It allows to

reduce the solution of control problem to finite-dimensional nonlinear moments problem. The procedure of derivation of moments problem is illustrated in details on the example of one-dimensional equation of thermal conductivity. The solution of obtained moments problem is found in a particular case. Based on obtained results a computer simulation is done using COMSOL Multiphysics platform in one-dimensional case for a rod. The main dependences of control function against input data of the problem are revealed. The state of the rod for several (constant) values of the source intensity is expressed in terms of graphs and illustrations. Corresponding illustrations are brought in case of control absence (null-power source) for comparison.

An effective numerical scheme for solving the obtained system of nonlinear constraints is suggested in the case of extended class of admissible controls. Calculation of control parameters is reduced to the simplest problem of nonlinear programming.

Introduction

The problem of bodies heating was always of interest for both engineers and theoreticians. In view of variety of possibilities to heat given body to a needed regime, as well as of amount of corresponding resources spent on the heating, a natural question arises about optimal choice of heating method. Optimality criterion may be different depending on a particular method: time optimality, heating with source of minimal intensity, heating with minimal quantity of sources, heating with optimal placements (fixed or changeable) of sources with given configuration, heating with optimal change of the external temperature field etc. Internal heat sources, heat sources with localized distribution on boundary of the heating body, heat sources with ability to change their localization in time discretely or continuously (scanning sources) can be used for this purpose. One may also fix the placements of the sources and carry out the heating by controlling the power of the source, however from control theory for distributed parameters systems point of view that problem does not have any significance. In [1] various interesting problems of plates and shells heating optimization are considered using methods of variational calculus.

In [2–4] a practical problem of bodies heating via electronic ray spread out from a point source, which can be freely moved over the body outer surface along controllable trajectory was stated by Russian scientist A. G. Butkovskiy and his colleagues. The problem was the characterization of trajectory for the source which provides required temperature distribution in the body when the power of the source is prescribed. Mathematically the problem was the determination of the control function included in the right-hand side of (linear) heat equation, naturally, in nonlinear manner. In order to solve the problem the Fourier method of variables separation was applied initially and it was reduced to an infinite-dimensional nonlinear problem of moments. Even though by that time the existence and uniqueness of optimal solution of the problem was proved [5], general algorithm for explicit solution construction was not developed yet. Further developments in this subject allowed not only describing such an algorithm [6], but to propose and develop other efficient approaches to solve mobile control problems as well. Namely, in articles [7, 8] two different methods of control by right-hand side of partial differential equations– *the method of substitution* and *the method of implementation*. Application of those methods in turn gives a unified approach for solving mobile control problems. Using the method of substitution first we need to determine such a distributed control on which the required state of controlled system is achieved and is held further. Then, using the method of implementation we are able to find unknown parameters of mobile control close in a certain sense to distributed control obtained earlier by the method of substitution. The algorithms of all methods described above are outlined in monograph [9] with many illustrative examples. Articles [10, 11], published after monograph [9], are devoted to generalization of substitution and implementation methods for two-dimensional systems, at this the considered heat equations is *nonlinear*.

In a recently published article by A. G. Butkovskiy [12] the mobile control problem is referred as one of the most important rigorously unsolved problems in control theory of

distributed parameter systems. Since the problem has significant practical value and, nevertheless, is rigorously unsolved, any approximate algorithm with easily implementable numerical scheme is very important and is required. Other mobile control problems for distributed parameter systems are considered, for example, in [13–17], mainly for vibrating systems with distributed parameters.

The Bubnov–Galerkin procedure [18], which initially was proposed for solving problems in elasticity theory, now is applied to solve boundary value problems arising in almost all areas of applied mathematics. In recent article [19] an algorithm is proposed for approximate solving control problems for bilinear systems. The solution is reduced to a finite-dimensional problem of moments resolvable explicitly. In material distribution, designs structure and topology optimization etc. problems, mathematically formulated via bilinear systems, in which the control function does not explicitly depend on one of independent variables (more often time-variable), application of Butkovskiy`s generalized method [20–25] and the Bubnov–Galerkin procedure in turn is suggested. The algorithm is applied in one [19, 26] and two-dimensional [27, 28] cases. Its numerical scheme was turned out to be very simple and easily implementable; solution of a nonlinear programming problem with equality and inequality type constraints is needed.

In this paper we suggest an approximate procedure for solving mobile (moving, scanning) control problems in dimensions three. The procedure is explained in details for spatial linear heat equation subjected to linear boundary conditions. The purpose of the problem is to explicitly construct a trajectory (motion law) for the heat source in order to provide required terminal temperature distribution when initial temperature distribution in the body is given. The solution of the (non-stationary) state equation is approximated via Bubnov–Galerkin procedure and a coupled system of Cauchy problems with respect to unknown coefficients of solution expansion is derived. The system is resolved explicitly. Force the approximating solution to satisfy prescribed terminal condition with required precision, for determination of the control function, a finite-dimensional system of necessary and sufficient conditions are obtained. This system is treated as nonlinear problem of moments. An example of nonlinear problem of moments derivation procedure is provided for one-dimensional heat equation. The solution of obtained problem is constructed in a particular case.

Based on obtained results a computational experiment is set up in order to reveal the sensitivity of a functional on control function (exactly its length) with respect to system external and internal parameters. Main results are discussed.

In order to derive a solution with easily implementable numerical scheme, the set of admissible controls is extended into the set of compactly supported Lebesgue measurable functions in order to accommodate sliding modes and the control function is represented as a piecewise constant function. Then, the determination of control trajectory parameters is reduced to a problem of nonlinear programming with constraints of equality and inequality types. Numerical solution of obtained problem is described through a table and graphs.

List of Notations

Throughout the paper the following notations and abbreviations:

$C_p\left(\mathcal{O}\right)$ – the space of piecewise-continuous in $\mathcal{O}$ functions,

$\mathcal{Q} = \left\{ q \in C_p[0,T]; \operatorname{supp} q = [0,T] \right\} -$ denotes any of the space of admissible controls $\mathcal{U}, \mathcal{V}$ или $\mathcal{W},$ and $q -$ any of control u, v or $w.$

$\operatorname{supp} \varphi = \overline{\left\{ x \in \mathbb{R}^n, \varphi(x) \neq 0 \right\}} -$ the support of function $\varphi(x),$

$\overline{\mathcal{O}} -$ the closure, and $\partial \mathcal{O} -$ the boundary of domain $\mathcal{O},$

$\nabla -$ the gradient,

$\mathcal{P} = \left\{ p \in C_p[0,T]; |p(t)| \leq 1, t \in [0,T] \right\},$

$\Theta(x, y, z, t) -$ the temperature distribution,

$\delta(x, y, z) = \delta(x)\delta(y)\delta(z) -$ the spatial Dirac function,

$\mathrm{T}_m(x) = \cos(m \arccos x) -$ Chebyshev polynomials of the first kind,

$\delta_m^\mu = \delta_\mu^m = \begin{cases} 1, & m = \mu, \\ 0, & m \neq \mu, \end{cases} -$ Kronecker`s symbol,

$\operatorname{sign} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases} -$ sign function,

$\theta(x) = \begin{cases} 1, & x > 0, \\ 0.5, & x = 0, \\ 0, & x < 0, \end{cases} -$ Heaviside unit step function,

the set $\{1, 2, ..., N\},$ for short, we denote by $\{1; N\},$

$c_p -$ heat capacity,

$\rho -$ density,

$\kappa -$ coefficient of thermal conductivity,

$\chi = \dfrac{\kappa}{c_p \rho} -$ thermal diffusivity.

1. Problem Statement

Let we are aimed to heat a solid occupying finite domain $\mathcal{O} \subset \mathbb{R}^3$ with piecewise smooth boundary $\partial \mathcal{O},$ into a prescribed heat-state. Let the boundary of the solid, for

instance, to be in heat exchange with external medium of constant (null) temperature[1]. To achieve the aim we are allowed to use a point heat source with intensity varying in time $p \in \mathcal{P}$, able to move over the surface of the solid along any prescribed piecewise-continuous trajectory

$$\mathbb{T} = \left\{ u \in \mathcal{U}, v \in \mathcal{V}, w \in \mathcal{W}; \operatorname{supp}(x-u), \operatorname{supp}(y-v), \operatorname{supp}(z-w) \subset \mathcal{O} \right\} \subset \mathcal{O},$$

$$\text{for all } t \in [0, T].$$

We have an opportunity to heat the body as by choosing a proper intensity law, as well as an easily realizable trajectory for the source.

The state of the solid satisfies heat equation [2–13]

$$c_p \rho \frac{\partial \Theta}{\partial t} = \nabla \left[\kappa \nabla \Theta \right] + p(t) \delta \big(x - u(t), y - v(t), z - w(t) \big), \tag{1}$$

$$(x, y, z, t) \in \mathcal{O} \times (0, T).$$

It is supposed, that the coefficients of (1) depends on all independent variables, at this $0 < c_p$, ρ, $\kappa \in C_p \big(\mathcal{O}; C[0,T] \big)$, $\kappa' \in C_p \big(\mathcal{O}; C[0,T] \big)$. The boundary conditions are written as follows

$$\Theta = 0, \qquad (x, y, z, t) \in \partial\mathcal{O} \times [0, T]. \tag{2}$$

Remark 1. The scanning source can also have other shapes. In general, it has to satisfy restrictions [2–11]

$$\varphi(x, y, z) \geq 0, \quad \operatorname{supp} \varphi \subset \mathcal{O}, \int_{\mathcal{O}} \varphi(x, y, z) \, dxdydz = 1.$$

Particular cases include cylindrical, elliptical, rectangular shapes [30, 31], and shapes described by Gauss function [9–11] etc.

Remark 2. As it turns out in practice, control by intensity of the source is not far easy and, moreover, very expensive in sense of resources spent. At the same time, control by motion (trajectory) of the source is easily implementable and does not cost so much. Moreover, there is a wide class of required heat regimes that are unreachable by appropriate choice of source power and can be achieved only by controlling its motion [9].

The initial temperature distribution in the body is supposed to be known:

$$\Theta(x, y, z, 0) = \Theta_0(x, y, z), \ (x, y, z) \in \overline{\mathcal{O}}. \tag{3}$$

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[1] Basically, this assumption does not play any significant role for applicability of suggesting method. In general, it can be replaced by a more practical assumption, however, as it will be seen below, the boundary conditions, mathematically expressing that assumption must be necessarily remain linear. Without losing the generality, the boundary conditions may be chosen to be homogeneous [1].

Our aim now can be expressed in terms of used notation. We are aimed to explicitly describe the set of all piecewise-continuous trajectories $\mathbb{T}$, ensuring for solution of initial-boundary value problem (1)–(3) the following terminal condition

$$\Theta(x,y,z,T) = \Theta_T(x,y,z), \ (x,y,z) \in \overline{\mathcal{O}}. \tag{4}$$

We will additionally assume that the power (intensity) of the source is given. We require also the consistency of boundary conditions with initial and terminal data.

2. Problem Solution

To solve the problem we are going to use the Bubnov–Galerkin procedure. Suppose that we were able to construct an orthonormal system of ansatz (approximating) functions $\{\sigma_m(x,y,z)\}_{m \in \mathbb{N}}$, satisfying to prescribed boundary conditions. Then, the approximating solution of (1) will be represented in the following form [18, 32]

$$\Theta_N(x,y,z,t) = \sum_{m=1}^{N} \omega_m(t)\sigma_m(x,y,z), \ (x,y,z,t) \in \overline{\mathcal{O}} \times [0,T], \tag{5}$$

where the coefficients $\omega_m(t)$, of the expansion, according to the Bubnov–Galerkin procedure, are obtained from the following Cauchy problem

$$\omega'_\mu(t) - \sum_{m=1}^{N} \mathcal{K}_m^\mu \omega_m(t) = \Omega_\mu(t), \tag{6}$$

$$\omega_\mu(0) = \Theta_{0\mu}, \tag{7}$$

$$\mu \in \{1; N\},$$

in which

$$\mathcal{K}_m^\mu = \frac{1}{c_p \rho} \int_{\mathcal{O}} \sigma_\mu(x,y,z) \nabla \left[\kappa \nabla \sigma_m(x,y,z) \right] dx\,dy\,dz,$$

$$\Omega_\mu(t) = \frac{p(t)}{c_p \rho} \int_{\mathcal{O}} \delta(x - u(t), y - v(t), z - w(t)) \sigma_\mu(x,y,z)\,dx\,dy\,dz =$$

$$= \frac{p(t)}{c_p \rho} \sigma_\mu(u(t), v(t), w(t)), \quad p \in \mathcal{P}.$$

Conditions (7) are obtained by taking into account the expansion of initial conditions into ansatz functions

$$\Theta_0(x,y,z) = \sum_{m=1}^{N} \Theta_{0m} \sigma_m(x,y,z).$$

Remark 3. In order to improve the convergence of Bubnov–Galerkin procedure, usually a system of weighted orthogonal functions is taken [31]. Especially, when the

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problem (1)–(3) is considered in a corresponding Sobolev space, in which Chebyshev polynomials of the first kind form an orthonormal basis, the most reasonable way of thinking might be the transformation of domain $\overline{\mathcal{O}}$ into cube $[-1,1] \times [-1,1] \times [-1,1]$, and as $\{\sigma_m(x,y,z)\}_{m \in \mathbb{N}}$ consideration of system $\{\mathrm{T}_m(x)\mathrm{T}_n(y)\mathrm{T}_k(z)\}_{m,n,k \in \mathbb{N}}$, orthonormal in $[-1,1] \times [-1,1] \times [-1,1]$, with weight $\left[(1-x^2)(1-y^2)(1-z^2)\right]^{-\frac{1}{2}}$.

Let us write the Cauchy problem (6), (7) in matrix form as follows

$$\omega' = \mathcal{K}\omega + \Omega, \tag{8}$$

$$\omega(0) = \Theta_0, \tag{9}$$

in which ω, Ω and Θ_0 denote vector columns consisting of unknown functions, right-hand sides and initial function (6), (7), respectively, and $\mathcal{K}$ denotes matrix consisting of coefficients of system (6). The general solution of Cauchy problem (8), (9) gives the Cauchy integral representation formula

$$\omega(t) = \Phi[t,0]\Theta_0 + \int_0^t \Phi[t,\tau]\Omega(\tau)\,d\tau, \tag{10}$$

in which $\Phi[t,\tau]$ is the (fundamental) Cauchy matrix of the system. Force function (10) to satisfy the expansion of terminal condition (4) with respect to chosen ansatz functions $\{\sigma_m(x,y,z)\}_{m \in \{1;N\}}$:

$$\Theta_T(x,y,z) = \sum_{m=1}^N \Theta_{Tm}\sigma_m(x,y,z),$$

in order to derive a system on necessary and sufficient conditions with respect to unknown functions in purpose of system controllability

$$\int_0^T \Phi[T,\tau]\Omega(\tau)\,d\tau = \Theta_T - \Phi[T,0]\Theta_0, \tag{11}$$

in which Θ_T is vector column consisting of coefficients of expansion of (4) mentioned above.

Remark 4. As a result of computation it might turn out that for some finite $N_0 \in \mathbb{N}$ approximating solution (5) is exact solution for the problem, i.e. the residue obtained by substitution of approximating solution (5) in equation (1) vanishes uniformly. Otherwise, the number N of ansatz functions in (5) can be chosen in a manner to provide a required precision. For instance, it can be chosen such that

$$\left\| \Theta_0 - \sum_{m=1}^N \Theta_{0m}\sigma_m \right\| \to \min, \quad \left\| \Theta_T - \sum_{m=1}^N \Theta_{Tm}\sigma_m \right\| \to \min,$$

in metrics of spaces of initial and terminal data. Error estimates for Bubnov–Galerkin procedure is carried out in [31].

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Note, that unknown controls also depends on choice of N so we should rather use symbols $u_N(t)$, $v_N(t)$ and $w_N(t)$. Howoever, to be short in text, we shall omit that fact and just keep it in mind throughout the whole paper.

The only parameter in system (11) for fixed T is the required trajectory, the components of which are figured out in the functional from left-hand side of the system. At the same time, its right-hand side is the value of that functional on a particular admissible trajectory. Namely, that admissible trajectory we are seeking for.

The components of required trajectory can be determined from system (11) in several manners. For instance, one may choose a cost functional having proper physical interpretation[2] for a particular problem and minimize it with respect to control functions which in their turn satisfy equality type restrictions (11). This problem can be solved through efficient numerical methods of nonlinear programming [33]. System (11) can be interpreted as a classical problem of moments [2–6, 9, 12] as well. In view of nonlinearity of inclusion of trajectory $\mathbb{T}$ into components of vector column $\Omega,$ the derived problem of moments, naturally, will be nonlinear.

3. The Nonlinear Problem of Moments in One-Dimensional Case

Let us illustrate the derivation scheme of system (11) in the simplest one-dimensional case for a rod with constant parameters. Let the object of heating is a sufficiently thin homogeneous rod thermo-isolated at edges. Figure 1 shows the computational model simulating the influence of a point heat source on the rod. Then, the solution of the corresponding one-dimensional heat equation will satisfy the following boundary conditions (after rescaling all variables and functions)

$$\Theta(-1,t) = \Theta(1,t) = 0, \ t \in [0,T],$$

The control function (in this case the trajectory has only one component) must satisfy to restriction $|u(t)| < 1$ for all $t \in [0,T]$.

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[2] As such a functional one may take, for instance

$$L[u,v,w] = \int_0^T \sqrt{\dot{u}^2(t) + \dot{v}^2(t) + \dot{w}^2(t)}\, dt,$$

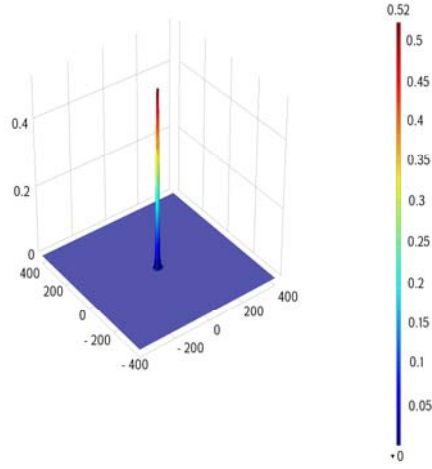which in our treatment characterizes the trajectory of the source.

Figure 1. The computational form of a point source.

Initial and terminal data are given through the following functions

$$\Theta(x,0) = \Theta_0(x), \quad \Theta(x,T) = \Theta_T(x), \ x \in [-1,1].$$

As system of ansatz functions in this case we may consider, for instance, the system of trigonometric sines $\{\sin(\pi m x)\}_{m \in \mathbb{N}}$ obviously satisfying prescribed boundary conditions. Then

$$\mathcal{K}_m^\mu = -\chi(\pi m)^2 \int_{-1}^{1} \sin(\pi m x) \sin(\pi \mu x) dx = -\chi(\pi m)^2 \delta_m^\mu,$$

$$\Omega_\mu(t) = \frac{p(t)}{c_p \rho} \int_{-1}^{1} \delta(x - u(t)) \sin(\pi \mu x) dx = \frac{p(t)}{c_p \rho} \sin(\pi \mu u(t)), \ p \in \mathcal{P}.$$

Instead of coupled system (6), (7) now for unknowns $\omega_\mu(t)$ we shall derive the following independent system of Cauchy problems

$$\omega'_\mu(t) + \chi(\pi \mu)^2 \omega_\mu(t) = \Omega_\mu(t), \tag{12}$$

$$\omega_\mu(0) = \Theta_{0\mu}, \tag{13}$$

$$\mu \in \{1; N\},$$

in which $\Theta_{0\mu}$ are the coefficients of the expansion of initial data with respect to chosen ansatz functions. The general solution of system (12), (13) reads as

$$\omega_\mu(t) = \left[\Theta_{0\mu} + \int_{0}^{t} \exp\left[\chi(\pi\mu)^2 \tau\right] \Omega_\mu(\tau) d\tau \right] \exp\left[-\chi(\pi\mu)^2 t\right], \ \mu \in \{1; N\}. \tag{14}$$

Force this function to satisfy to expansion of terminal condition with respect to chosen ansatz functions, we will arrive at

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$$\int_0^T \exp\left[\chi(\pi\mu)^2 \tau\right] \Omega_\mu(\tau) d\tau = \Theta_{T\mu} \exp\left[\chi(\pi\mu)^2 T\right] - \Theta_{0\mu}, \quad \mu \in \{1; N\},$$

which coincides with corresponding moment equalities from [2–6, 9, 12].

After determination of required trajectory $u(t)$ from that system, the approximating solution of corresponding one-dimensional problem will be

$$\Theta_N(x,t) = \sum_{\mu=1}^N \left[\Theta_{0\mu} + \int_0^t \exp\left[\chi(\pi\mu)^2 \tau\right] \Omega_\mu(\tau) d\tau\right] \exp\left[-\chi(\pi\mu)^2 t\right] \sin(\pi\mu x),$$

$$(x,t) \in [-1,1] \times [0,T].$$
(15)

4. Example

Let the initial temperature distribution in the rod considered in the previous section is given as follows $\Theta(x,0) = \sin(\pi x)$, $x \in [-1,1]$, and we are required to heat the rod before some fixed T into a state

$$\Theta(x,T) = \begin{cases} \Theta_T = \text{const}, & x \in (-1,1), \\ 0, & x = \pm 1. \end{cases}$$

The power of the source is constant: $p(\tau) = p_0 = \text{const}$. Let us first determine the coefficients of the expansion of initial and terminal functions into chosen ansatz functions. It is obvious that

$$\Theta_{0\mu} = \delta_\mu^1, \quad \Theta_{T\mu} = \frac{2\Theta_T}{\pi\mu}\left[1 - (-1)^\mu\right],$$

therefore from (14) we will have

$$\int_0^T \exp\left[\chi(\pi\mu)^2 \tau\right] \sin(\pi\mu u(\tau)) d\tau = \mathcal{M}_\mu, \quad \mu \in \{1; N\},$$
(16)

$$\mathcal{M}_\mu = \frac{c_p\rho}{p_0}\Theta_{T\mu} \exp\left[\chi(\pi\mu)^2 T\right] - \frac{c_p\rho}{p_0}\delta_\mu^1, \quad \mu \in \{1; N\}.$$

It is quite easy to see that $\mathcal{M}_{2\mu} = 0$, $\mathcal{M}_\mu = O(\mu^{-1})$, $\mu \to \infty$. Figure 2 shows the discrete dependence $\mathcal{M}_\mu \leftrightarrow \mu$, $\mu \in \{1; 200\}$, which is almost the same for all values of parameters taken.

For determination of control function from (16), we will first use the algorithm for solving nonlinear problem of moments suggested in [6, 9]. Let us introduce the expression

$$\Lambda(l_1,l_2,...,l_N,u) = \int_0^T \max_u \lambda_N(l_1,l_2,...,l_N,\tau,u) d\tau, \quad |u(\tau)| < 1,$$

in which

$$\lambda_N \left(l_1, l_2, ..., l_N, \tau, u \right) = \sum_{\mu=1}^{N} l_\mu \exp \left[\chi \left(\pi \mu \right)^2 \tau \right] \sin \left(\pi \mu u \left(\tau \right) \right)$$

l_μ, $\mu \in \{1; N\}$, are undetermined multipliers, at this

$$\sum_{\mu=1}^{N} l_\mu^2 > 0.$$

Then, we calculate the derivative of expression $\lambda_N \left(l_1, l_2, ..., l_N, \tau, u \right)$ with respect to u and find its roots. As a result we get

$$\frac{\partial \lambda_N}{\partial u} = \pi \sum_{\mu=1}^{N} \mu l_\mu \exp \left[\chi \left(\pi \mu \right)^2 \tau \right] \cos \left(\pi \mu u \left(\tau \right) \right) = 0.$$
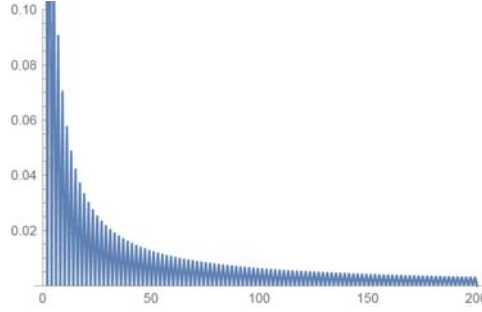
Figure 2. The discrete dependence $\mathcal{M}_\mu \leftrightarrow \mu$.

The analytical exact solution of derived functional equation for any $N \in \mathbb{N}$ is very hard to achieve. For fixed $N \in \mathbb{N}$ it can be done using the system of symbolic computation Wolfram Mathematica. It has the simplest form for $N = 2$:

$$u_{1,2} \left(\tau \right) = \pm \frac{1}{\pi} \arccos \left[\sqrt{\eta^2 \left(\tau \right) + \frac{1}{2}} - \eta \left(\tau \right) \right],$$

$$u_{3,4} \left(\tau \right) = \pm \frac{1}{\pi} \arccos \left[-\sqrt{\eta^2 \left(\tau \right) + \frac{1}{2}} - \eta \left(\tau \right) \right], \quad \tau \in [0, T],$$

$$\eta \left(\tau \right) = \frac{l_1}{8 l_2} \exp \left[-3 \chi \pi^2 \tau \right] \geq -\frac{1}{4}, \quad \tau \in [0, T].$$

The last equality follows from the fact that the control trajectory must be real-valued. It is easy to see that the functional $\Lambda \left(l_1, l_2, u \right)$ does not satisfy condition of theorem 2 from [9] for all l_1, l_2. In such cases according to [9] one is recommended to transform the moment equalities into the following ones

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$$\int_0^T \exp\left[\chi\pi^2\tau\right]\left[\sin\left(\pi u\left(\tau\right)\right)-\gamma\right]d\tau = \mathcal{M}_1 - \gamma\frac{\exp\left[\chi\pi^2 T\right]-1}{\chi\pi^2} := \mathcal{M}_1',$$

$$\int_0^T \exp\left[\chi\left(2\pi\right)^2\tau\right]\sin\left(2\pi u\left(\tau\right)\right)d\tau = 0,$$

in which $\gamma = \mathrm{const}$ is chosen in such a way that the new functional $\Lambda\left(l_1,l_2,u\right)$ becomes positive for all l_1, l_2.

After some algebraic rearrangements, one can prove that

$$\mathrm{sign}\frac{\partial^2\lambda_2}{\partial u^2} = \mp\mathrm{sign}\left[l_1\left(1\pm\sqrt{1+\frac{1}{2\eta^2}}\right)\right],$$

thus for $\mathrm{sign}\, l_1 = 1$ we have

$$\lambda_2\left(l_1,l_2,\tau,u_1\right) = \min_u\lambda_2\left(l_1,l_2,\tau,u\right),\ \ \lambda_2\left(l_1,l_2,\tau,u_4\right) = \min_u\lambda_2\left(l_1,l_2,\tau,u\right),$$

$$\lambda_2\left(l_1,l_2,\tau,u_2\right) = \max_u\lambda_2\left(l_1,l_2,\tau,u\right),\ \ \lambda_2\left(l_1,l_2,\tau,u_3\right) = \max_u\lambda_2\left(l_1,l_2,\tau,u\right),$$

and for $\mathrm{sign}\, l_1 = -1-$

$$\lambda_2\left(l_1,l_2,\tau,u_1\right) = \max_u\lambda_2\left(l_1,l_2,\tau,u\right),\ \ \lambda_2\left(l_1,l_2,\tau,u_4\right) = \max_u\lambda_2\left(l_1,l_2,\tau,u\right),$$

$$\lambda_2\left(l_1,l_2,\tau,u_2\right) = \min_u\lambda_2\left(l_1,l_2,\tau,u\right),\ \ \lambda_2\left(l_1,l_2,\tau,u_3\right) = \min_u\lambda_2\left(l_1,l_2,\tau,u\right).$$

Moreover, it is easy to check that

$$\lambda_2\left(l_1,l_2,\tau,u_1\right) = -\lambda_2\left(l_1,l_2,\tau,u_2\right),\ \ \lambda_2\left(l_1,l_2,\tau,u_3\right) = -\lambda_2\left(l_1,l_2,\tau,u_4\right).$$

It is necessary and sufficient for solvability of system (16) fulfilment of [6, 9]

$$\min_{l_1,l_2}\Lambda\left(l_1,l_2,u\right) \geq 1,\ \ \text{with}\ \ l_1\mathcal{M}_1' + l_2\mathcal{M}_2 = 1. \tag{17}$$

Since in our particular case $\mathcal{M}_2 = 0$, therefore l_1 is immediately defined. Substituting it into expression of $\Lambda\left(l_1,l_2,u\right)$, for solvability of problem of moments (16) we derive

$$\min_{l_2}\Lambda\left(\frac{1}{\mathcal{M}_1'},l_2,u\right) \geq 1,$$

where l_2 may be computed from by efficient numerical methods of nonlinear programming [33]. If for some l_2^o in the last inequality the equality sign takes place, then for almost every $\tau\in\left[0,T\right]$ the solution of (16) satisfies the following maximum condition [6, 9]

$$\lambda_2\left(\frac{1}{\mathcal{M}_1'}, l_2^o, \tau, u^o\right) = \max_u \lambda_2\left(\frac{1}{\mathcal{M}_1'}, l_2^o, \tau, u\right).$$

In order to define an initial approximation to start computing l_2^o one may use the restriction

$4\eta(\tau) \geq -1$, $\tau \in [0, T]$, from where it follows that $l_2 \geq -\dfrac{\exp\left[-3\chi\pi^2 T\right]}{2\mathcal{M}_1'}$. Then,

$-\dfrac{\exp\left[-3\chi\pi^2 T\right]}{2\mathcal{M}_1'}$ will be the required initial approximation for computing l_2^o.

Computations are done for various values of dimensionless parameters

$$\alpha^2 = \chi\frac{T_*}{l^2 T} \equiv \frac{\kappa}{c_p \rho}\frac{T_*}{l^2 T} \text{ and } p_0 = p_*\frac{\Theta_*}{l c_p \rho}\frac{T_*}{T},$$

in which l is the half-length of the rod, T_* is the prescribed heating time, p_* is the intensity of the source, and Θ_* is a scaling intensity for temperature distribution in the rod.

Figure 3 expresses the dependence $\Lambda\left(l_1, l_2, u^o\right) \leftrightarrow l_2$ for $\chi = 1.11\cdot 10^{-4}$ m²/sec (copper) and $T = 2\pi$. In this case $l_2 = 0.088$, $\min\limits_{l_2} \Lambda\left(l_1, l_2, u^o\right) = 3.97 > 1$. If for fixed γ, l_1 is positive, the minimum in (17) is attained with $u^o = u_4(t)$, otherwise — with $u^o = u_2(t)$, at this in both cases l_2^o is equal to its first approximation.
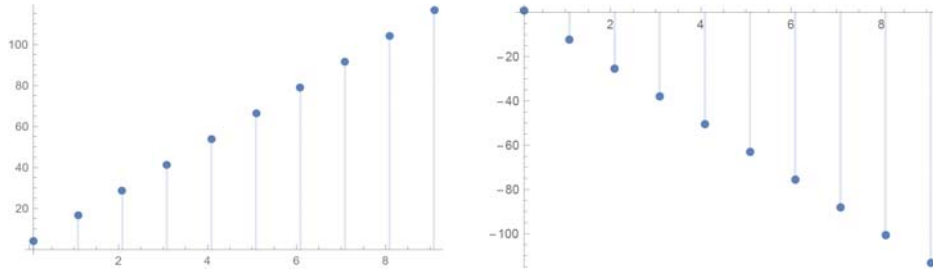

Figure 3. The dependence $\Lambda\left(l_1, l_2, u_4\right) \leftrightarrow l_2$ (left) and $\Lambda\left(l_1, l_2, u_1\right) \leftrightarrow l_2$ (right) for a rod made from copper: $\chi = 1.11\cdot 10^{-4}$ m²/sec.

In order to reveal the sensitivity of control function with respect to other external and internal parameters of the system, we have computed the length of trajectory u^o for various combinations of those parameters. It turned out, that the control function is most sensitive with respect to rod length, at this for fixed length of the rod, the length of control trajectory does not sufficiently depend on rod material (computations are done for rods made from copper, aluminum, silver and steel, for which χ has almost the same order).

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With increase (decay) of time T_* when the intensity of the source is fixed, the quantity $L[u_*]$ irregularly decreases (increases). It turned out as well that the numerical scheme of this algorithm requires high computational cost.

5. Numerics

On the basis of obtained results a computational experiment is set up, the main results of which are expressed graphically on Figures 4–11. The rod has length 2 m, and its cross sectional area (square) is equal to $1/40$ m². Physical characteristics of the rod are identical to those of copper. Figure 4 (left) shows the computational analogue of the rod, which is divided into 140 parts (elements), due to which its degrees of freedom (DOF) are equal to 4542 (2427 of them are internal). Figure 4 (right) shows so-called convergence plot, characterizing the reciprocal of the time step size versus the time step.
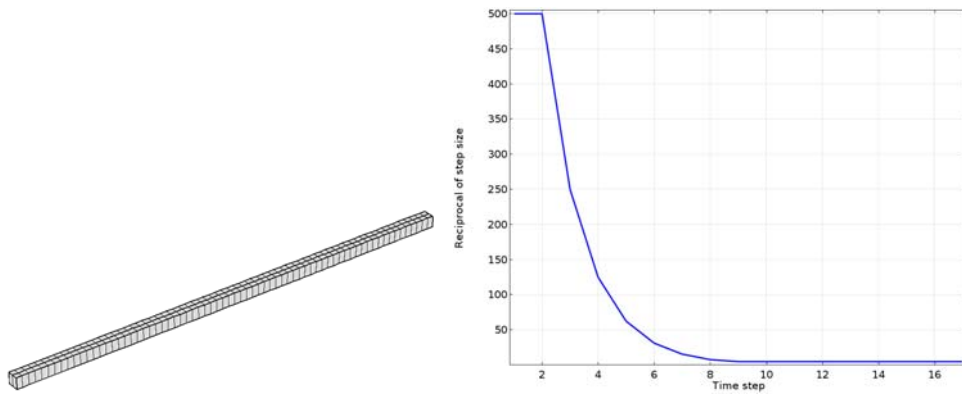
Figure 4. Computational analogue of the rod (left) and convergence plot (right).

On Figure 5 the temperature distribution along the rod length is plotted on fixed time moment $t_* = 250$ sec for four particular values of the source intensity p_*. Figure 6 shows the temperature distribution in spatial model of the rod for $p_* = 0$ and $t_* = 0$ (left), $t_* = 250$ sec (right). Figure 7 shows the temperature distribution in spatial model of the rod for $p_* = 7.5$ J/sec and $t_* = 0$ (left), $t_* = 250$ sec (right).
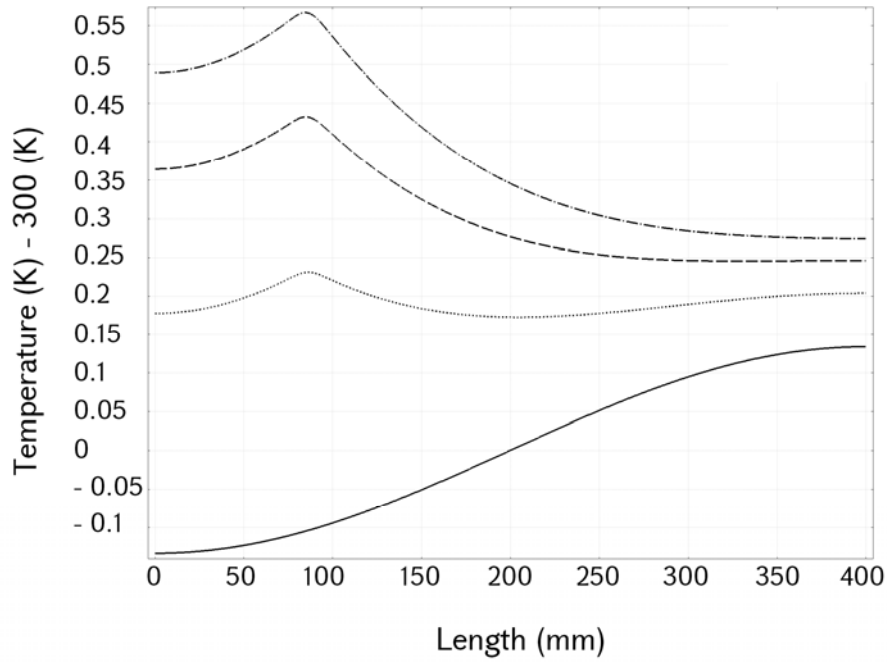
Figure 5. Temperature distribution along the rod length for $t_* = 250$ sec and different intensities of the source: $p_* = 0$ (solid line), $p_* = 7.5$ J/sec (dotted line), $p_* = 12$ J/sec (dashed line), $p_* = 15$ J/sec (dash-dot line).

On Figure 8 the temperature distribution along the rod length is plotted on fixed time moment $t_* = 350$ sec for four particular values of the source intensity p_*. Figure 9 shows the temperature distribution in spatial mod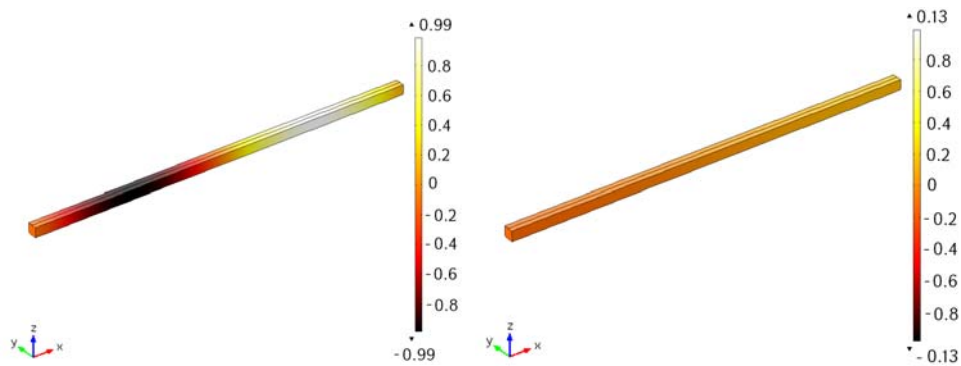el of the rod for $p_* = 0$ (left), $p_* = 3$ J/sec (right). On Figure 10 the temperature distribution along the rod length is plotted on fixed time moment $t_* = 450$ sec for four particular values of the source intensity p_*. Figure 9 shows the temperature distribution in spatial model of the rod for $p_* = 0$ (left), $p_* = 1.4$ J/sec (right).

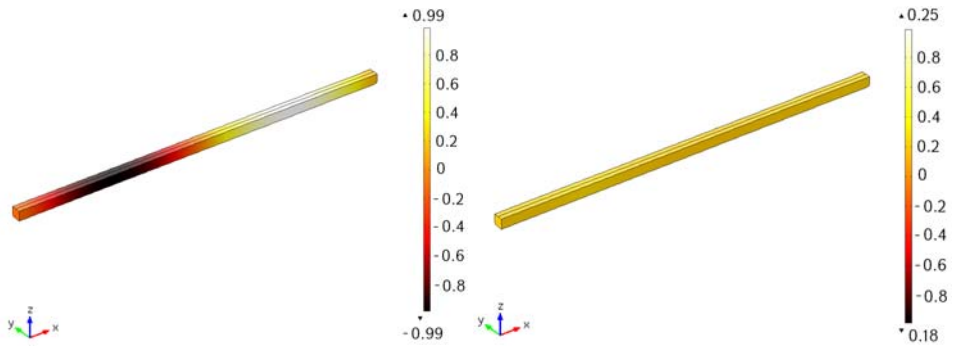**Figure 6.** The state of the rod for $p_* = 0$ $t_* = 0$ (left) and $t_* = 250$ sec (right).

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Figure 7. The state of the rod for $p_* = 7.5$ J/sec, $t_* = 0$ (left) and $t_* = 250$ sec (right).
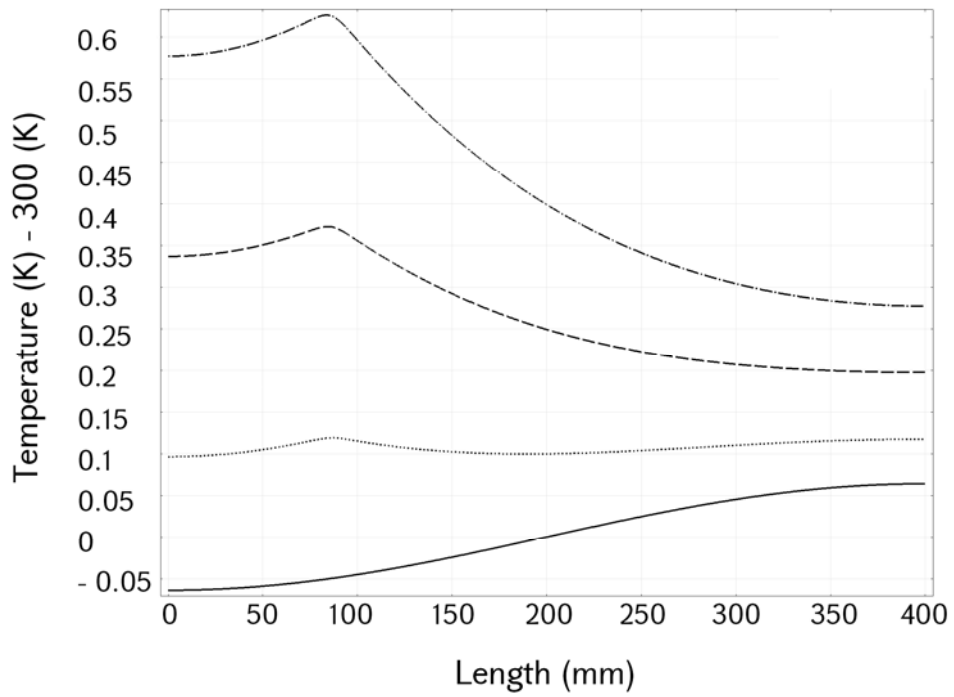

Figure 8. Temperature distribution along the rod length for $t_* = 350$ sec and different intensities of the source: $p_* = 0$ (solid line), $p_* = 3$ J/sec (dotted line), $p_* = 7.5$ J/sec (dashed line), $p_* = 12$ J/sec (dash-dot line).
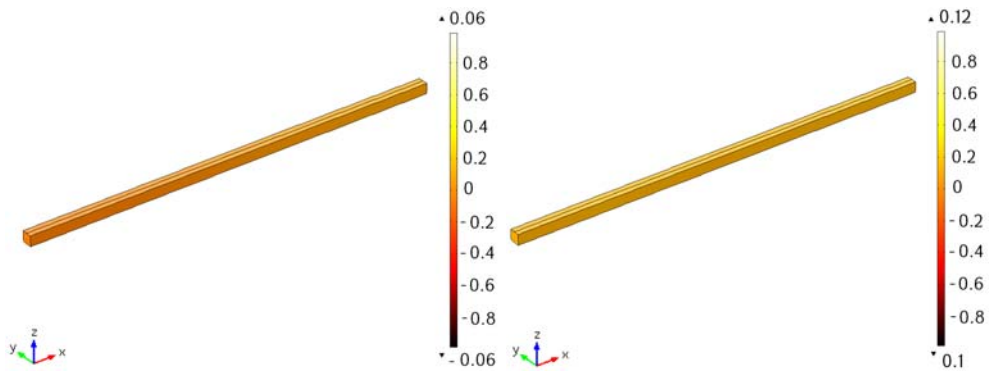
Figure 9. The state of the rod for $t_* = 350$ sec, $p_* = 0$ (left), $p_* = 3$ J/sec (right).
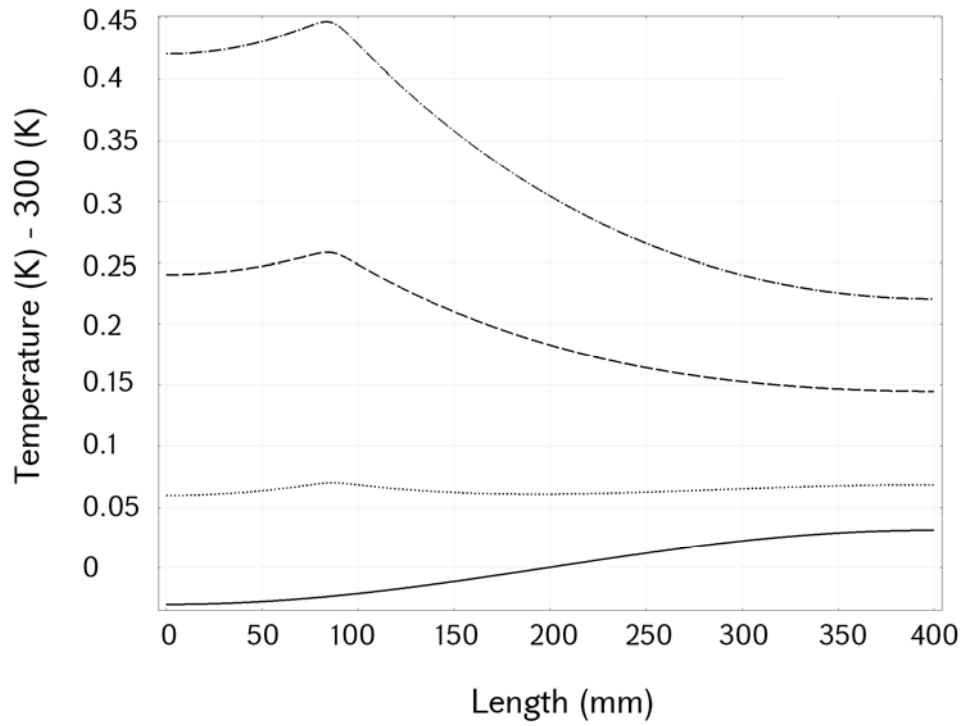

Figure 10. Temperature distribution along the rod length for $t_* = 450$ sec and different intensities of the source: $p_* = 0$ (solid line), $p_* = 1.4$ J/sec (dotted line), $p_* = 4.2$ J/sec (dashed line), $p_* = 7$ J/sec (dash-dot line).
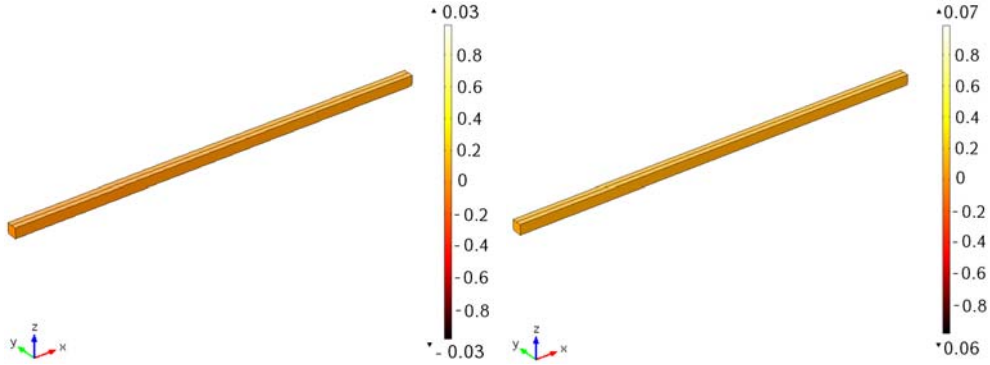
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Figure 11. The state of the rod for $t_* = 450$ sec, $p_* = 0$ (left), $p_* = 1.4$ J/sec (right).

6. Introduction of Sliding Modes

As it was mentioned above, the realization of numerical scheme of the algorithm chosen for solving nonlinear problem of moments requires high computational costs. The situation gets even worse for large N; the number of required operations, and therefore computational time, essentially increases. Besides, there is a wide class of initial and required heating regimes that are unreachable by piecewise-continuous trajectories of heat the source with prescribed intensity. However, it turns out [9], that it can be circumvented by extending the set of admissible controls including in it sliding modes. Let us extend the set of admissible controls to the set

$$\mathcal{U} = \left\{ u \in L^1[0,T]; \ \operatorname{supp} u \subseteq [0,T], \ |u| < 1 \right\},$$

and to represent the trajectory of the source as a piecewise-constant function [20, 25, 27, 28, 34]

$$u(t) = \sum_{k=1}^{M} u_k \left[\theta(t - t_{k-1}) - \theta(t - t_k) \right], \quad t \in [0,T], \tag{18}$$

uniquely defined through unknown parameters u_k and t_k, $k \in \{1; M\}$, which have to be found from the following system of equality type constraints

$$\frac{1}{\chi(\pi\mu)^2} \sum_{k=1}^{M} \left[\exp\left[\chi(\pi\mu)^2 t_k \right] - \exp\left[\alpha(\pi\mu)^2 t_{k-1} \right] \right] \sin(\pi\mu u_k) = \mathcal{M}_\mu, \tag{19}$$

$$\mu \in \{1; N\}.$$

This *finite dimensional discrete* system is obtained by substitution of (18) into (16). Note also, that in general the control (18) is discontinuous in switching points t_k, $k \in \{1; M\}$, where it has second order discontinuities.

The problem of computation of unknowns u_k and t_k, $k \in \{1; M\}$, now can be stated in terms of nonlinear programming, taking functional

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$$L[u] = \sum_{k=1}^{M} |u_k|,$$

characterizing the length of required trajectory, as cost functional. To system of equality type constraints (19) one has to add the following restrictions of inequality type

$$|u_k| < 1, \quad 0 \le t_{k-1} < t_k \le T, \; k \in \{1; M\}.$$

Opportunities of numerical minimization of Wolfram Mathematica 10 package allows to solve that problem very efficiently and without too much computational costs even fo M and N large enough (using the built-in operators NMinimize, FindMinimum, NMinValue and FindMinValue). For solving this problem numerically with precision ε only $O(\varepsilon^{-1})$ operations are needed [33]. In Table 1 parameters of control function are presented for various combinations of ratio Θ_0/p_0, coefficient α^2 and parameter T. For instance, when $\Theta_0/p_0 = -0.2$, $\alpha^2 = 4.7 \cdot 10^{-4}$, $T = 4\pi$, the control function takes the form

$$u(t) = 0.99 \big[\theta(t - 0.) - \theta(t - 2.81) + \theta(t - 2.82) - \theta(t - 12.55) \big] +$$
$$+ 0.43 \big[\theta(t - 12.55) - \theta(t - 12.56) \big] +$$
$$+ 0.82 \big[\theta(t - 12.56) - \theta(t - 4\pi) \big], \qquad t \in [0, 4\pi].$$

The existence of solution to the problem is checked in traditional manners [33]. Corresponding restrictions obtained as a result will allow us to underline the boundary of application of suggested scheme.

Θ_0/p_0	$\alpha^2 \cdot 10^4$	T	u_k	t_k
0.25	3.86	2π	$u_1 = u_3 = u_5 = u_6 = 0.,$ $u_2 = 0.78, \, u_4 = 0.58$ $u_7 = 0.66$	$t_1 = t_2 = 0.41,$ $t_3 = t_4 = 0.62,$ $t_5 = 1.36, \, t_6 = 1.94$
-0.2	4.37	4π	$u_1 = 0., \, u_4 = 0.38,$ $u_6 = 0.43, \, u_7 = 0.82,$ $u_2 = u_3 = u_5 = 0.99$	$t_1 = 0., \, t_2 = 0.03,$ $t_3 = 2.81, \, t_4 = 2.82,$ $t_5 = 12.55, \, t_6 = 12.56$
0.5	1.38	4π	$u_2 = 0.76, \, u_3 = 0.15,$ $u_6 = 0.49, \, u_7 = 0.7,$ $u_1 = u_5 = u_6 = 0$	$t_1 = 0.24, \, t_2 = 0.41,$ $t_3 = 0.62, \, t_4 = 0.65,$ $t_5 = 1.27, \, t_6 = 1.92$

Table 1. Parameters of control function for M=10.

Besides providing an efficient numerical scheme, function (18) can be easily carried out in practice and has a simple physical interpretation; it characterizes piecewise-constant

trajectory for the source, namely in whole time-interval $\left[t_{k-1}, t_k\right]$ the source heats the point $x = u_k$ of the rod.

Conclusion

In this article we propose a new approximate algorithm for solving problems of mobile control for systems with distributed parameters. The algorithm is based on the Bubnov-Galerkin procedure and is set up on example of three-dimensional heat equation with parameters varying in time and spatial coordinates. Mathematical statement of the problem requires determination of a control function included in right-hand side of the governing equation in general nonlinearly. The Bubnov-Galerkin procedure allows to reduce the problem to a finite dimensional system of equality type constraints treated here as classical problem of moments. Numerical simulations done on the basis of obtained formulas for one-dimensional thin rod reveal the main dependences of control function from all other external and internal parameters of the system. A computational experiment is set up for thin rod made from copper, the main results of which are brought graphically.

It was observed that the method of solution chosen requires high computational costs and thus not efficient. In order to get an efficient numerical scheme the set of admissible controls is extended into set of Lebesgue measurable functions compactly supported in boundary of considered domain allowing to use sliding modes. The unknown function is represented as piecewise constant (discontinuous in general), the parameters of which are to be computed from a problem of nonlinear programming under equality and inequality type constraints.

One of the algorithm privileges is that it also can be applied for numerical solution of mobile control problems for *nonlinear* or other type of state equations (including integral, integro-differential, differential-difference etc.), equations with variable coefficients, particularly, for wave equation, for coupled systems of mentioned equations, and for sources of other forms as well. One of essential disadvantages of the method is its applicability only in the case of linear boundary conditions.

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