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GOODNESS-OF-FIT TESTS FOR CONTINUOUS-TIME STATIONARY PROCESSES

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Abstract. The paper considers the following problem of hypotheses testing: based on a finite realization $\{X(t),\, 0 \le t \le T\}$ of a zero mean real-valued mean square continuous stationary Gaussian process $X(t)$, $t \in \mathbb{R}$, construct goodness-of-fit tests for testing a hypothesis H_0 that the hypothetical spectral density of the process $X(t)$ has the specified form. We show that in case where the hypothetical spectral density of $X(t)$ does not depend on unknown parameters (the hypothesis H_0 is simple), then the suggested test statistic has a chi-square distribution. In the case where the hypothesis H_0 is composite, that is, the hypothetical spectral density of $X(t)$ depends on an unknown p–dimensional vector parameter, we choose an appropriate estimator for unknown parameter and describe the limiting distribution of the test statistic, which is similar to that of obtained by Chernov and Lehman in the case of independent observations. The testing procedure works both for short- and long-memory models.[1]

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1. Introduction

Suppose we observe a finite realization $\mathbf{X}_T := \{X(t),\, 0 \le t \le T\}$ of a zero mean real-valued mean square continuous stationary Gaussian process $X(t)$, $t \in \mathbb{R}$, possessing a spectral density function.

In this paper, we consider the following problem of hypotheses testing: based on a sample $\mathbf{X}_T$ construct goodness-of-fit tests for testing a hypothesis H_0 that the spectral density function of the process $X(t)$ has the specified form. We will distinguish the following two cases:

a) The hypothesis H_0 is simple, that is, the hypothetical spectral density $f(\lambda)$ of $X(t)$ does not depend on unknown parameters.
b) The hypothesis H_0 is composite, that is, the hypothetical spectral density $f(\lambda)$ of $X(t)$ depends on an unknown p–dimensional vector parameter $\theta =$

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$(\theta_1, \ldots, \theta_p)' \in S$, that is, $f(\lambda) = f(\lambda, \theta)$, $\lambda \in \mathbb{R}$, $\theta \in S$, where S is an open set of Euclidean space $\mathbb{R}^p$.

We first consider the relatively easy case a) of simple hypothesis H_0. Denote by $I_T(\lambda)$ the continuous periodogram (the empirical spectral density) of the process $X(t)$:

$$I_T(\lambda) = \frac{1}{2\pi T} \left| \int_0^T X(t) e^{-it\lambda} \, dt \right|^2. \tag{1.1}$$

To test the hypothesis H_0, it is natural to introduce a measure of divergence (disparity) of the hypothetical and empirical spectral densities, and construct a goodness-of-fit test based on the distribution of the chosen measure. There are different type of such measures of divergence.

In this paper, as a measure of divergence of the hypothetical spectral density $f(\lambda)$ and empirical spectral density $I_T(\lambda)$, we consider the m–dimensional random vector

$$\Phi_T = (\Phi_{1T}, \ldots, \Phi_{mT}) \tag{1.2}$$

with elements

$$\Phi_{jT} := \Phi_{jT}(\mathrm{X}_T) = \frac{\sqrt{T}}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \left[\frac{I_T(\lambda)}{f(\lambda)} - 1 \right] \varphi_j(\lambda) d\lambda, \quad j = 1, 2, \ldots, m, \tag{1.3}$$

where $\{\varphi_j(\lambda),\, j = 1, 2, \ldots, m\}$ is some orthonormal system on $\mathbb{R}$:

$$\int_{-\infty}^{+\infty} \varphi_k(\lambda) \varphi_j(\lambda) \, d\lambda = \delta_{kj}. \tag{1.4}$$

We show that under wide conditions on $f(\lambda)$ and $\varphi_j(\lambda)$, the random vector Φ_T has asymptotically $N(0, I_m)$–normal distribution as $T \to \infty$, where I_m is $m \times m$ identity matrix, and the components Φ_{kT} and Φ_{jT} are asymptotically uncorrelated for $k \neq j$. Therefore in the case of simple hypothesis H_0, we can use the statistics

$$S_T = S_T(\mathrm{X}_T) := \Phi_T'(\mathrm{X}_T) \Phi_T(\mathrm{X}_T) = \sum_{j=1}^{m} \Phi_{jT}^2(\mathrm{X}_T), \tag{1.5}$$

which for $T \to \infty$ will have a χ^2 distribution with m degrees of freedom.

Thus, fixing an asymptotic level of significance α we can consider the class of goodness-of-fit tests for testing the simple hypothesis H_0 about the form of the spectral density f with asymptotic level of significance α determined by critical regions of the form $\{\mathrm{x}_T : S_T(\mathrm{x}_T) > d_\alpha\}$, where $S_T(\mathrm{x}_T)$ is given by (1.5), and d_α is the α-quantile of χ^2–distribution with m degrees of freedom, that is, d_α is determined from the condition

$$P(\chi^2 > d_\alpha) = \int_{d_\alpha}^{\infty} k_m(x) \, dx = \alpha, \tag{1.6}$$

where $k_m(x)$ is the density of χ^2–distribution with m degrees of freedom.

In the case b) of composite hypothesis H_0, that is, when the hypothetical spectral density function $f(\lambda, \theta)$ of the underlying process $X(t)$ depends on an unknown p-dimensional parameter $\theta = (\theta_1, \ldots, \theta_p)'$, the problem of construction of goodness-of-fit tests becomes more complex, because first the unknown parameter has to be estimated. It is important to remark that in this case the limiting distribution of the test statistic will change in accordance with properties of an estimator of θ, and generally will not be a χ^2–distribution.

For testing a composite hypothesis H_0, we again can use statistics of type (1.5), but with a statistical estimator $\widehat{\theta}_T$ instead of unknown θ. The corresponding statistics can be written as follows:

$$S_T(\widehat{\theta}_T) = S_T(\mathbf{X}_T, \widehat{\theta}_T) := \Phi_T'(\mathbf{X}_T, \widehat{\theta}_T)\Phi_T(\mathbf{X}_T, \widehat{\theta}_T) = \sum_{j=1}^{m} \Phi_{jT}^2(\widehat{\theta}_T), \tag{1.7}$$

where now

$$\Phi_T(\mathbf{X}_T, \widehat{\theta}_T) := \left(\Phi_{1T}(\mathbf{X}_T, \widehat{\theta}_T), \ldots, \Phi_{mT}(\mathbf{X}_T, \widehat{\theta}_T)\right) \tag{1.8}$$

with elements

$$\Phi_{jT}(\mathbf{X}_T, \widehat{\theta}_T) := \frac{\sqrt{T}}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \left[\frac{I_T(\lambda)}{f(\lambda, \widehat{\theta}_T)} - 1\right] \varphi_j(\lambda) d\lambda, \quad j = 1, 2, \ldots, m. \tag{1.9}$$

So, we must choose an appropriate statistical estimator $\widehat{\theta}_T$ for unknown θ, and determine the limiting distribution of statistic (1.7). Then, having the limiting distribution of statistic (1.7), for given level of significance α we can consider the class of goodness-of-fit tests for testing the composite hypothesis H_0 about the form of the spectral density f with asymptotic level of significance α determined by critical regions of the form

$$\{\mathbf{x}_T : \; S_T(\mathbf{x}_T, \widehat{\theta}_T) > d_\alpha\}, \tag{1.10}$$

where d_α is the α-quantile of the limiting distribution of the statistic (1.7), that is, d_α is determined from the condition

$$\int_{d_\alpha}^{\infty} \widehat{k}_m(x)\, dx = \alpha, \tag{1.11}$$

where $\widehat{k}_m(x)$ is the density of the limiting distribution of $S_T(\widehat{\theta}_T)$ defined by (1.7).

The limiting distribution of statistic (1.5) for discrete-time Gaussian stationary processes was considered by Hannan [9]. For independent observations the limiting distributions of statistics of type (1.7) with various statistical estimators $\widehat{\theta}_T$ have been considered by many authors (see, e.g., Chernov and Lehman [2], Chibisov [3], Cramer [4], Kendall and Stuart [10], Dzhaparidze and Nikulin [14], and references therein). For observations generated by discrete-time short-memory Gaussian stationary processes the limiting distribution of statistics (1.7) for different statistical estimators $\widehat{\theta}_T$ of unknown parameter θ has been studied by Dzhaparidze [5] and Osidze [11], [12]. (Recall that a stationary processes $X(t)$ is of short-memory if the spectral density $f(\lambda)$ of $X(t)$ is bounded away from zero and infinity, that is, there are constants C_1 and C_2 such that $0 < C_1 \leq f(\lambda) \leq C_2 < \infty$.) In the case where the spectral density $f(\lambda)$ has zeros and/or poles, the limiting distribution of statistics (1.7) for discrete-time processes has been described in Ginovyan [8].

The present paper extends some results of the above cited references to the continuous-time case and for a broader class of spectral densities possibly possessing zeros and poles.

The rest of the paper is organized as follows: In Section 2 we state the main results of the paper   Theorems 2.1 and 2.2. In Section 3 we present some auxiliary results. Section 4 contains proofs of Theorems 2.1 and 2.2.

2. The main results

We first introduce some notation, definitions and assumptions.

Given numbers $p \geq 1$, $0 < \alpha < 1$, $r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of natural numbers, we set $\beta = \alpha + r$ and denote by $H_p(\beta)$ the L^p-Hölder class, that is, the class of those functions $\psi(\lambda) \in L^p(\mathbb{R})$, which have r-th derivatives in $L^p(\mathbb{R})$ and with some positive constant C satisfy

$$||\psi^{(r)}(\cdot + h) - \psi^{(r)}(\cdot)||_p \leq C|h|^\alpha.$$

Definition 2.1. *We say that a pair of integrable functions $(f(\lambda), g(\lambda))$, $\lambda \in \mathbb{R}$, satisfies condition $(\mathcal{H})$, and write $(f,g) \in (\mathcal{H})$, if $f(\lambda) \in H_p(\beta_1)$ for $\beta_1 > 0$, $p > 1$ and $g(\lambda) \in H_q(\beta_2)$ for $\beta_2 > 0$, $q > 1$ with $1/p + 1/q = 1$, and one of the conditions a) – d) is fulfilled:*

a) $\beta_1 > 1/p$, $\beta_2 > 1/q$,

b) $\beta_1 \leq 1/p$, $\beta_2 \leq 1/q$ and $\beta_1 + \beta_2 > 1/2$,

c) $\beta_1 > 1/p$, $1/q - 1/2 < \beta_2 \leq 1/q$,

d) $\beta_2 > 1/q$, $1/p - 1/2 < \beta_1 \leq 1/p$.

The next theorem contains sufficient conditions for statistic S_T, given by (1.5), to have a limiting (as $T \to \infty$) χ^2–distribution with m degrees of freedom, extending the result stated in Hannan [9] (p. 94).

Theorem 2.1. *Let the spectral density $f(\lambda)$ and the orthonormal functions $\{\varphi_j(\lambda), j = 1, 2, \ldots, m\}$ be such that $(f, g_j) \in (\mathcal{H})$ for all $j = 1, 2, \ldots, m$, where $g_j = \varphi_j/f$.*

Then the limiting (as $T \to \infty$) distribution of statistics $S_T = S_T(\mathbf{X}_T)$ given by (1.5) is a χ^2-distribution with m degrees of freedom.

Now we consider the case of composite hypothesis H_0, and assume that the hypothetical spectral density $f = f(\lambda, \theta)$ is known with the exception of a vector parameter $\theta = (\theta_1, \dots, \theta_p) \in \Theta \subset \mathbb{R}^p$. In order to construct the corresponding test, we first have to choose an appropriate statistical estimator $\widehat{\theta}_T$ for the unknown parameter θ, constructed on the basis of a sample $\mathbf{X}_T = \{X(t),\, 0 \le t \le T\}$.

Let us introduce the following set of assumptions:

A1) The true value θ_0 of the parameter θ belongs to a bounded closed set Θ contained in an open set S in the p-dimensional Euclidean space $\mathbb{R}^p$.

A2) If θ_1 and θ_2 are two distinct points of Θ, then $f(\lambda, \theta_1) \neq f(\lambda, \theta_2)$ almost everywhere in $\mathbb{R}$ with respect to the Lebesgue measure.

A3) For $\theta \in \Theta$, $(f, g_j) \in (\mathcal{H})$ for all $j = 1, 2, \dots, m$, where $f = f(\lambda, \theta)$ and $g_j = \varphi_j(\lambda)/f(\lambda, \theta)$.

A4) For $\theta \in \Theta$, $(f, h_{kj}) \in (\mathcal{H})$ for all $k = 1, 2, \dots, p$ and $j = 1, 2, \dots, m$, where $f = f(\lambda, \theta)$ and $h_{kj} = \dfrac{\varphi_j(\lambda)}{f(\lambda, \theta)} \dfrac{\partial}{\partial \theta_k} \ln f(\lambda, \theta)$.

A5) The $(p \times p)$–matrix $\Gamma(\theta_0) = ||\gamma_{kj}(\theta_0)||_{k,j=\overline{1,p}}$ with elements

$$\gamma_{kj}(\theta_0) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial \theta_k} \ln f(\lambda, \theta)\right]_{\theta=\theta_0} \left[\frac{\partial}{\partial \theta_j} \ln f(\lambda, \theta)\right]_{\theta=\theta_0} d\lambda \tag{2.1}$$

is nonsingular.

A6) There exists a $\sqrt{T}$–consistent estimator $\widehat{\theta}_T$ for the parameter θ such that the following asymptotic relation holds:

$$\sqrt{T}\,(\widehat{\theta}_T - \theta_0) - \Gamma^{-1}(\theta_0)\Delta_T(\theta_0) = o_P(1), \tag{2.2}$$

where $\Gamma^{-1}(\theta_0)$ is the inverse of the matrix $\Gamma(\theta_0)$ defined in A5), $\Delta_T(\theta) = (\Delta_{1T}(\theta), \dots, \Delta_{pT}(\theta))$ is a p-dimensional random vector with components

$$\Delta_{kT}(\theta) = \frac{T^{1/2}}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \left[\frac{I_T(\lambda)}{f(\lambda, \theta)} - 1\right] \frac{\partial}{\partial \theta_k} \ln f(\lambda, \theta)\, d\lambda, \quad k = \overline{1, p}, \tag{2.3}$$

and the term $o_P(1)$ tends to zero in probability as $T \to \infty$. (Recall that an estimator $\widehat{\theta}_T$ for θ is said to be $\sqrt{T}$–consistent if $\sqrt{T}(\widehat{\theta}_T - \theta)$ is bounded in probability).

Let $B(\theta) = ||b_{jk}(\theta)||_{j=\overline{1,m},\, k=\overline{1,p}}$, be a $(m \times p)$–matrix with elements

$$b_{jk}(\theta) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \varphi_j(\lambda) \frac{\partial}{\partial \theta_k} \ln f(\lambda, \theta)\, d\lambda, \tag{2.4}$$

where $\varphi_j(\lambda)$ $(j = \overline{1, m})$ are the functions from (1.3).

Theorem 2.2. *Under the assumptions A1)–A6) the limiting distribution (as $T \to \infty$) of the statistics $S_T(\mathbf{X}_T, \widehat{\theta}_T)$ given by (1.7), coincides with the distribution of the random variable*

$$\sum_{j=1}^{m-p} \xi_j^2 + \sum_{j=1}^{p} \nu_j \, \xi_{m-p+j}^2, \tag{2.5}$$

where ξ_j, $j = \overline{1,m}$, are iid $N(0,1)$ random variables, while the numbers ν_k ($0 \le \nu_k < 1$), $k = \overline{1,p}$, are the roots relative to ν of equation

$$\det\left[(1-\nu)\Gamma(\theta_0) - B'(\theta_0)\, B(\theta_0)\right] = 0. \tag{2.6}$$

Remark 2.1. For independent observations the result of Theorem 2.2 was first obtained by Chernov and Lehman [2] (see, also, Chibisov [3]). For observations generated by discrete-time short-memory Gaussian stationary processes the result was stated by Osidze [11], [12] (see, also, Dzhaparidze [5]). In the case where the spectral density has zeros and/or poles, the result for discrete-time processes was proved by Ginovyan [8]. Note also that for continuous-time processes with rational spectral densities Theorem 2.2 under more restrictive assumptions was stated by Osidze [11], [12].

3. Lemmas

Given a number $T > 0$ and an integrable real symmetric function $h(\lambda)$ defined on $\mathbb{R}$, the *T-truncated Toeplitz operator* generated by $h(\lambda)$, denoted by $\mathbb{W}_T(h)$, is defined by the following equation (see, e.g., [6]):

$$[\mathbb{W}_T(h)u](t) = \int_0^T \hat{h}(t-s)u(s)ds, \quad u(s) \in L^2[0,T], \tag{3.1}$$

where $\hat{h}(\cdot)$ is the Fourier transform of $h(\cdot)$:

$$\hat{h}(t) = \int_{-\infty}^{+\infty} e^{it\lambda} h(\lambda) d\lambda.$$

For the proof of the next two lemmas we refer to [6], [7].

Lemma 3.1. *Let $h_i(\lambda)$, $i = 1, 2, \ldots, m$, be integrable real symmetric functions defined on $\mathbb{R}$ such that $h_i \in L^1(\mathbb{R}) \cap L^{p_i}(\mathbb{R})$, $p_i > 1$, $i = 1, 2, \ldots, m$, with $1/p_1 + \ldots + 1/p_m \le 1$. Then the following limiting relation holds:*

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{tr}\left[\prod_{i=1}^m \mathbb{W}_T(h_i)\right] = (2\pi)^{m-1} \int_{-\infty}^{\infty} \left[\prod_{i=1}^m h_i(\lambda)\right] d\lambda, \tag{3.2}$$

where $\operatorname{tr}[A]$ stands for the trace of an operator A.

Lemma 3.2. *Let $f(\lambda)$ be the spectral density of the process $X(t)$ and $g(\lambda)$ be an integrable real symmetric function defined on $\mathbb{R}$ such that $(f,g) \in (\mathcal{H})$. Let $I_T(\lambda)$ be the periodogram of $X(t)$ given by (1.1). Then the random variable*

$$(3.3) \qquad \eta_T = T^{1/2} \int_{-\infty}^{\infty} [I_T(\lambda) - f(\lambda)]\, g(\lambda)\, d\lambda$$

has asymptotically (as $T \to \infty$) normal distribution with 0 mean and variance σ^2 :

$$(3.4) \qquad \sigma^2 = 4\pi \int_{-\infty}^{\infty} f^2(\lambda)\, g^2(\lambda)\, d\lambda.$$

Below we will use the following well-known result, which is known as the Cramér-Wold device (or theorem) (see [1], Theorem 29.4).

Lemma 3.3 (Cramér-Wold device). *For random vectors $\mathbf{X}_n = (X_{n1}, \ldots, X_{nk})$ and $\mathbf{X} = (X_1, \ldots, X_k)$ a necessary and sufficient condition for $\mathbf{X}_n \xrightarrow{d} \mathbf{X}$ is that $\sum_{j=1}^k t_j X_{nj} \xrightarrow{d} \sum_{j=1}^k t_j X_j$ for each $\mathrm{t} = (t_1, \ldots, t_k) \in \mathbb{R}^k$, where $\xrightarrow{d}$ stands for convergence in distribution.*

Consider the $(m+p)$–dimensional random vector column $\Psi_T(\theta) = (\Phi_T(\theta), \Delta(\theta))$, where $\Phi_T(\theta)$ is defined by (1.2), (1.3), while $\Delta(\theta)$ is as in assumption A6).

Using Cramér-Wold device, as an immediate consequence of Lemma 3.2 we can state the following result.

Lemma 3.4. *Under the assumptions of Theorem 2.2, the random vector $\Psi_T(\theta) = (\Phi_T(\theta), \Delta(\theta))$ has asymptotically $N(0, G(\theta_0))$ - normal distribution as $T \to \infty$ with*

$$G(\theta_0) = \begin{pmatrix} I_m & B(\theta_0) \\ B'(\theta_0) & \Gamma(\theta_0) \end{pmatrix},$$

where I_m is $m \times m$ identity matrix, and $B(\theta)$ is the $(m \times p)$–matrix with elements given by (2.4).

Lemma 3.5. *Let $\widehat{\theta}_T$ be a $\sqrt{T}$–consistent estimator for unknown parameter θ. Then under the assumptions of Theorem 2.2, for $T \to \infty$ the following asymptotic relation holds:*

$$(3.5) \qquad \Phi_T(\widehat{\theta}_T) = \Phi_T(\theta_0) - \sqrt{T} B(\theta_0)\,(\widehat{\theta}_T - \theta_0) + o_P(1),$$

where B is defined by (2.4), and term $o_P(1)$ tends to zero in probability as $T \to \infty$.

Proof. Using the mean value theorem we can write

$$\Phi_{jT}(\widehat{\theta}_T) - \Phi_{jT}(\theta_0) = \frac{\sqrt{T}}{\sqrt{4\pi}} \int_{-\infty}^{\infty} I_T(t) \left[\frac{1}{f(t, \widehat{\theta}_T)} - \frac{1}{f(t, \theta_0)} \right] \varphi_j(t)\, dt$$

$$(3.6) \qquad = \frac{\sqrt{T}}{\sqrt{4\pi}} \sum_{k=1}^{p} (\widehat{\theta}_{kT} - \theta_{k0}) \int_{-\infty}^{\infty} I_T(t) \left[\frac{1}{f(t,\theta)} \frac{\partial}{\partial \theta_k} \ln f(t,\theta) \right]_{\theta=\theta_*} \varphi_j(t)\, dt,$$

where $\theta_* \in (\theta_0, \widehat{\theta}_T)$. Since $\widehat{\theta}_T$ is a $\sqrt{T}$–consistent estimator for θ to complete the proof of (3.5), it is enough to show that as $T \to \infty$ we have

$$(3.7) \qquad \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} I_T(t) \left[\frac{1}{f(t,\theta)} \frac{\partial}{\partial \theta_k} \ln f(t,\theta) \right]_{\theta=\theta_*} \varphi_j(t)\, dt = b_{jk}(\theta_0) + o_P(1),$$

where b_{jk} are as in (2.4).

Denote $g_* = g_*(t) = \left[\frac{1}{f(t,\theta)} \frac{\partial}{\partial \theta_k} \ln f(t,\theta) \right]_{\theta=\theta_*} \varphi_j(t)$ and $f_0 = f(t,\theta_0)$. Using Lemma 3.1 with $m = 2$, $h_1 = f_0$ and $h_2 = g_*$, for the expectation of the random variable on the left-hand side of (3.7), we obtain as $T \to \infty$

$$\mathbb{E}\left[\frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} I_T(t) \left[\frac{1}{f(t,\theta)} \frac{\partial}{\partial \theta_k} \ln f(t,\theta) \right]_{\theta=\theta_*} \varphi_j(t) = \frac{1}{\sqrt{4\pi}} \frac{1}{4\pi T} \operatorname{tr}\left[\mathbb{W}_T(f_0) \mathbb{W}_T(g_*) \right] \right.$$

$$(3.8) \qquad \longrightarrow \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \varphi_j(\lambda) \frac{\partial}{\partial \theta_k} \ln f(\lambda,\theta)\, d\lambda = b_{jk}(\theta_0).$$

Next, using Lemma 3.1 with $m = 4$, $h_1 = h_2 = f_0$ and $h_3 = h_4 = g_*$, for the variance of the random variable on the left-hand side of (3.7), we obtain as $T \to \infty$

$$(3.9) \qquad \mathbb{D}\left[\frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} I_T(t) \left[\frac{1}{f(t,\theta)} \frac{\partial}{\partial \theta_k} \ln f(t,\theta) \right]_{\theta=\theta_*} \varphi_j(t)\, dt \right]$$

$$(3.10) \qquad = \frac{1}{16\pi^3 T^2} \operatorname{tr}\left[(\mathbb{W}_T(f_0) \mathbb{W}_T(g_*))^2 \right] \longrightarrow 0.$$

Now (3.7) follows from (3.8), (3.9) and Chebyshev inequality. Lemma 3.5 is proved. The result that follows is well-known (see [2], [3]).

Lemma 3.6. *Assume that a random vector $\eta_T = (\eta_{1T}, \ldots, \eta_{nT})$ has limiting $N(0, A)$ normal distribution (as $T \to \infty$). Then the limiting distribution of the random variable $\eta_T' \eta_T = \sum_{j=1}^{n} \eta_{jT}^2$ coincides with the distribution of $\sum_{j=1}^{n} \lambda_j \xi_j^2$, where ξ_j, $j = \overline{1,n}$, are iid $N(0,1)$ random variables, while the numbers λ_j $(j = \overline{1,n})$ are the eigenvalues of matrix A. In particular, if A is an idempotent matrix, that is, $A^2 = A$, then $\eta_T' \eta_T$ has limiting χ^2–distribution with $k = \operatorname{tr}[A]$ degrees of freedom, where $\operatorname{tr}[A]$ stands for the trace of matrix A.*

4. Proofs

Proof of Theorem 2.1. The result immediately follows from Lemma 3.2 and the definition of χ^2-distribution. Indeed, applying Lemma 3.2 with $g_j = f/\varphi_j$, where φ_j, $j = 1, 2, \ldots, m$, satisfy (1.4), and using Cramér-Wold device (Lemma 3.3), we conclude that the random vector Φ_T, given by (1.2) and (1.3), has asymptotically $N(0, I_m)$-normal distribution as $T \to \infty$, where I_m is $m \times m$ identity matrix, and the components Φ_{kT} and Φ_{jT} are asymptotically uncorrelated for $k \neq j$. Therefore, the result follows from (1.5) and the definition of χ^2-distribution. □

Proof of Theorem 2.2. By (2.2) and Lemma 3.5, for $T \to \infty$ we have the asymptotic relation

$$\begin{aligned} \Phi_T(\widehat{\theta}_T) &= \Phi_T(\theta_0) - B(\theta_0)\sqrt{T}(\widehat{\theta}_T - \theta_0) + o_P(1) \\ &= \Phi_T(\theta_0) - B(\theta_0)\Gamma^{-1}(\theta_0)\Delta_T(\theta_0) + o_P(1). \end{aligned} \tag{4.1}$$

The last relation can be written in the form

$$\Phi_T(\widehat{\theta}_T) = U_T(\theta_0) + [V_T(\theta_0) - W_T(\theta_0)] + o_P(1), \tag{4.2}$$

where

$$U_T(\theta_0) = A(\theta_0)\Phi_T(\theta_0), \quad A(\theta_0) = I_m - B(\theta_0)(B'(\theta_0)B(\theta_0))^{-1}B'(\theta_0), \tag{4.3}$$

$$V_T(\theta_0) = B(\theta_0)\big(B'(\theta_0)B(\theta_0)\big)^{-1}B'(\theta_0)\Phi_T(\theta_0), \tag{4.4}$$

$$W_T(\theta_0) = B(\theta_0)\Gamma^{-1}(\theta_0)\Delta_T(\theta_0). \tag{4.5}$$

It is easy to see that

$$A(\theta_0)B(\theta_0) = 0. \tag{4.6}$$

Hence $U_T'(\theta_0)V_T(\theta_0) = U_T'(\theta_0)W_T(\theta_0) = 0$. Therefore, by (4.2)

$$\begin{aligned} \Phi_T'(\widehat{\theta}_T)\Phi_T(\widehat{\theta}_T) &= U_T'(\theta_0)U_T(\theta_0) + \\ &+ [V_T(\theta_0) - W_T(\theta_0)]'[V_T(\theta_0) - W_T(\theta_0)] + o_P(1). \end{aligned} \tag{4.7}$$

It follows from Lemma 3.4 and (4.6) that

$$\mathbb{E}[\mathrm{U_T}(\theta_0)(\mathrm{V_T}(\theta_0) - \mathrm{W_T}(\theta_0))'] \longrightarrow 0 \quad \text{as } T \to \infty.$$

Hence the terms on the right-hand side of (4.7) are asymptotically independent random variables. Next, it is easy to check that the matrix $A(\theta_0)$ in (4.3) is idempotent $A^2(\theta_0) = A(\theta_0)$ and $\mathrm{tr}[A(\theta_0)] = m - p$. Applying Lemmas 3.4 and 3.6 we conclude that the random variable $U_T'(\theta_0)U_T(\theta_0)$ has limiting χ^2–distribution with $m - p$ degrees of freedom.

To describe the limiting distribution of the second term on the right-hand side of (4.7), we first observe that by Lemma 3.4

$$\mathbb{E}[(\mathrm{V_T}(\theta_0) - \mathrm{W_T}(\theta_0))(\mathrm{V_T}(\theta_0) - \mathrm{W_T}(\theta_0))'] \longrightarrow \mathrm{B}(\theta_0)\left[(\mathrm{B}'(\theta_0)\mathrm{B}(\theta_0))^{-1} - \Gamma^{-1}(\theta_0)\right]\mathrm{B}'(\theta_0) \tag{4.8}$$

as $T \to \infty$. Therefore, by Lemma 3.6 the limiting distribution of the random variable

$$[V_T(\theta_0) - W_T(\theta_0)]\,[V_T(\theta_0) - W_T(\theta_0)]'$$

coincides with the distribution of the sum $\sum_{j=1}^{p} \nu_j \xi_{m-p+j}^2$, where ξ_j, $j = \overline{1,m}$, are iid $N(0,1)$ random variables, while the numbers ν_k $(k = \overline{1,p})$ are the non-zero eigenvalues of the matrix on the right-hand side of (4.8). By Lemma 4.3 from [3] the numbers ν_k $(k = \overline{1,p})$ coincide with the nonzero eigenvalues of the matrix $B'(\theta_0)B(\theta_0)\left[(B'(\theta_0)B(\theta_0))^{-1} - \Gamma^{-1}(\theta_0)\right]$, that is, ν_k are the roots relative to ν of equation

$$\det\left[B'(\theta_0)B(\theta_0)\left[(B'(\theta_0)B(\theta_0))^{-1} - \Gamma^{-1}(\theta_0)\right] - \nu I_p\right] = 0. \tag{4.9}$$

Since the matrix $\Gamma(\theta_0)$ is non-singular (4.9) is equivalent to (2.6).

To show that $0 \le \nu_k < 1$ for $k = \overline{1,p}$, first observe that by Lemma 3.4

(4.10)

$$\mathbb{E}\left[(\Delta_T(\theta_0) - B'(\theta_0)\Phi_T(\theta_0)\right]\left[\Delta_T(\theta_0) - B'(\theta_0)\Phi_T(\theta_0)\right]' \longrightarrow \Gamma(\theta_0) - B'(\theta_0)B(\theta_0)$$

as $T \to \infty$. Hence the matrix $\Gamma(\theta_0) - B'(\theta_0)B(\theta_0)$ is nonnegative definite. Therefore $(1-\nu)\Gamma(\theta_0) - B'(\theta_0)B(\theta_0) > 0$ for $\nu < 0$. On the other hand, since $\Gamma(\theta_0) > 0$ and $B'(\theta_0)B(\theta_0) > 0$ we have $(1-\nu)\Gamma(\theta_0) - B'(\theta_0)B(\theta_0) < 0$ for $\nu \ge 1$. Thus, $0 \le \nu_k < 1$. □

Список литературы

[1] P. Billingsley, Probability and Measure (3 ed.), John Wiley & Sons, New York (1995).

[2] H. Chernov, E. L. Lehman, "The use of maximum likelihood estimates in χ^2-tests for goodness of fit", Ann. Math. Statist., **25**, no. 3, 579 – 586 (1954).

[3] D. M. Chibisov, "Some tests of chi-square type for continuous distributions", Theory Probab. Appl., **16**, no. 1, 1 – 20 (1971).

[4] H. Cramer, Mathematical Methods of Statistics, Princeton University Press, Princeton (1946).

[5] K. Dzhaparidze, Parameter Estimation and Hypotesis Testing in Spectral Analysis of Stationary Time Series, Springer-Verlag, New York (1986).

[6] M. S. Ginovian, "On Toeplitz type quadratic functionals in Gaussian stationary process", Probab. Th. Rel. Fields, **100**, 395 – 406 (1994).

[7] M. S. Ginovian, "Asymptotic properties of spectrum estimate of stationary Gaussian processes", Journal of Contemporary Math. Anal., **30**, no. 1, 1 – 16 (1995).

[8] M. S. Ginovyan, "Chi-square type goodness-of-fit tests for stationary Gaussian process", Journal of Contemporary Math. Anal., **38**, no. 2, 1 – 13 (2003).

[9] E. J. Hannan, Time Series Analysis, John Wiley, New York (1960).

[10] M. G. Kendall and A. Stuart, The Advanced Theory of Statistics, **2**, (Inference and Relationship), Charles Griffin & Comp., London (1967).

[11] A. G. Osidze, "On a goodness of fit test in the case of dependence of spectral density of Gaussian processes on unknown parameters", Reports of AN Georgian SSR, **74**, no. 2, 273 – 276 (1974).

[12] A. G. Osidze, "On a statistic for testing the composite hypotesis regarding the form of a spectral density of a stationary Gaussian random process", Reports of AN Georgian SSR, **77**, no. 2, 313 – 315 (1975).

[13] K. O. Dzaparidze, "On the Estimation of the Spectral Parameters of a Gaussian Stationary Process with Rational Spectral Density", Theory Probab. Appl., **15**, no. 3, 531 – 538 (1970).

[14] K. O. Dzaparidze and M. S. Nikulin, "On a Modification of the Standard Statistics of Pearson", Theory Probab. Appl., **19**, no. 4, 851 – 853 (1975).

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